SOLUTIONS TO HANDBOOK PROBLEMS

The solutions provided here are only possible solutions. It is very likely that you or your students will come up with additional—and perhaps more elegant—solutions. Happy solving!

Warm-Up 1

1. There are 12 months in each year, so there are $12 \times 35 = 420$ months in 35 years.

2. If the function were $y = x$, the $y$ values would be the same as the $x$ values. In this function, the $y$ values are all 3 more than the $x$ values, so the function is $y = x + 3$, and $a = 3$.

3. There are $158 - 20 = 138$ numbers from 21 to 158 inclusive. Half of them, or 69 numbers, are odd.

4. Since the figure shows a square, $x - 2$ must equal 3. This is the case if $x = 5$.

5. If 20% of the grapes are red, then $100 - 20 = 80\%$ are not red. To find 80% of 50 grapes, we calculate $0.8 \times 50$ or $4/5 \times 50$. Either way, we get 40 grapes that are not red.

6. If the radius of the circle is 4 ft, then the diameter is 8 ft. This diameter is equal to the side length of the square, so the area of the square is $8 \times 8 = 64$ ft$^2$.

7. If the height of the similar triangle is double that of the original triangle, the base must also be double. With twice the height and twice the base, the larger triangle will have $2 \times 2 = 4$ times the area, or $4 \times 120 = 480$ mm$^2$.

8. Ten of the 16 books are math books, so the probability that the first book selected is a math book is $10/16 = 5/8$.

9. The volume of a prism is given by the formula $V = l \times w \times h$. We know the volume, the length and the width, so we have $120 = 5 \times 4 \times h$, to which the solution is $h = 120 ÷ 20 = 6$ in.

10. The prime factorization of 96 is $2 \times 2 \times 2 \times 2 \times 2 \times 3 = 2^5 \times 3$. The greatest of these primes is 3.

Warm-Up 2

11. The value of $2 \times (3 + 4)$ is $2 \times 7 = 14$, and the value of $2 \times 3 + 4$ is $6 + 4 = 10$. The positive difference is $14 - 10 = 4$.

12. For each of the 5 shirts Manny can wear, he can wear 3 different pairs of pants, which makes $5 \times 3 = 15$ combinations of shirts and pants. For each of these 15 combinations, he can wear either of 2 ties, which makes $15 \times 2 = 30$ outfits so far. Finally, when he picks one of 4 pairs of shoes, he gets one of $30 \times 4 = 120$ possible uniforms.

13. Each slice has a central angle of $360 ÷ 12 = 30$ degrees. The 10 remaining slices combined have a central angle of $30 \times 10 = 300$ degrees.

14. We are told that $78 = 2w + 6$. Solving this equation, we see that the width of the tennis court is $(78 - 6) ÷ 2 = 72 ÷ 2 = 36$ ft.

15. The “part-to-part” ratio 2:3 represents 5 parts in all. Sam should divide $2500 by 5 to find that each part is $500. Then he can calculate that Charity A gets $2 \times $500 = $1000, and Charity B gets $3 \times $500 = $1500$.

16. The odd prime numbers among the first 20 positive integers (counting numbers) are 3, 5, 7, 11, 13, 17 and 19. That’s 7 out of 20, so the probability is $7/20 = 7 ÷ 20 = 35\%$.

17. The value of $5 \times (11 + 4 ÷ 4)$ is $5 \times (11 + 1) = 5 \times 12 = 60$.

18. Since the two decagons (10 sides each) are similar, the perimeters should be in the same ratio as the corresponding sides. Thus the perimeter of the larger decagon is $76 \times 2 = 152$ cm.

19. The man will reach a location 50 mi north of his starting location on the day that he begins at a location 47 mi north of his starting location. After each day’s work and each night’s loss, the man gains only 1 mi per day, so it will take 47 days and 47 nights to wake up in the morning at mile 47. The next day he will paddle 3 mi north and first reach the 50-mile mark. So, it will take the man 48 days to first reach the 50-mile mark.

20. A regular hexagon can be divided into six equilateral triangles, as shown in the figure. The length of segment AD is twice the side length of the hexagon, so it’s $2 \times 6 = 12$ cm.
Exercise 1

21. From 10:30 to 12:00 is 90 minutes. Because 55 minutes have already passed, 35 minutes remain.

22. The sum of all six items is $27.14. It might be simplest to look for two items that add up to $27.14 – $17.36 = $9.78. The wallet at $3.49 and the puzzle at $6.29 add up to $9.78, so the desired difference is $6.29 – $3.49 = $2.80.

23. When the 5 × 5 × 5 cube is painted all of the exterior unit cubes will have at least one face painted, while the interior cubes will have no faces painted. So, inside of the 5 × 5 × 5 cube is a 3 × 3 × 3 cube composed of 27 unit cubes that did not receive any red paint.

24. The total area of the 4-ft-by-12-ft garden is the sum of its length and width, or 4 + 12 = 16 ft.

25. Suppose for a moment that we consider 000 a three-digit number. Then there are 10 numbers with a zero in the tens place (000, 101, 202, ... 909) that satisfy the given condition. There are only 9 such numbers with a one in the tens place (110, 211, 312, ... 918). Then there are 8 numbers with a two in the tens place (220, 321, 422, ... 927). This pattern continues until there is only 1 such number with a nine in the tens place (990). Thus, there would be 10 × 9 + 8 × 7 + 6 × 5 + 4 × 3 + 2 = 55 numbers, except that 000 is not really a three-digit number. Excluding this one number, there are 54 integers that satisfy the conditions.

26. We need to find 4/7 of 1421 dentists. It’s easier to find 1/7 of 1421 first, which is 1421 ÷ 7 = 203, and then multiply this by 4, which is 4 × 203 = 812 dentists.

27. If the width were 3 times as great, then we would have a square with an area of 3 × 48 = 144 ft². Since 12² = 144, the length would be 12 ft.

28. Fifteen dozen cookies is 3 times as many cookies as 5 dozen, so Amy will need 3 times as much flour. That’s 3 × 4 cups = 12 cups.

29. The volume of a rectangular prism is the product of its length, its width and its height. If we think of the length and the width in terms of the height, we have $l = 3h$ and $w = 2h$, and we know that the volume of 48 in³ is $3h × 2h × h = 6h³$. If $48 = 6h³$, then $h³ = 8$ and $h = 2$ in. The length is 3 times the height, so the length is $3 × 2 = 6$ in.

30. The store’s cost per can of soda is $7.68 ÷ 24 = $0.32. If the store is to make a profit of at least $4.40 per case, then it needs to charge at least $4.40 + $0.32 = $4.72 per case. The price per can must be at least $4.72 ÷ 24 = $0.19 per can above cost. The price per can must be at least $0.32 + $0.19 = $0.51.

Warm-Up 3

31. Hexagons have six sides, and regular means all six sides have the same length, so the perimeter is $6 × 6 = 36$ cm.

32. Runner A took 4 hours. Runners B and E took longer, and runners C and D took less time to run the marathon, so 4 hours was the median time of the five runners.

33. The greatest two-digit prime is 97 and the least two-digit prime is 11. Their product is $97 × 11 = 1067$.

34. Five cars is 5/3 times as much as 3 cars, so it should take $(5/3) × 4$ hours = 20/3 hours. To convert this to minutes, we multiply by 60 minutes per hour, which gives us $(20/3) × 60 = 20 × 20 = 400$ minutes.

35. There are a total of $2 + 3 + 4 + 3 = 12$ cars in the parking lot. Three are green and 9 are not green, so the probability that a car chosen at random is not green is $9/12 = 3/4$.

36. There are 36 in per yard, so Becca has $15 × 36 = 540$ in of ribbon. Dividing this into 20 equal parts, we find that each craft project can get $540 ÷ 20 = 27$ in of ribbon.

37. Division and multiplication have the same priority, so we calculate from left to right as follows: $2 ÷ 4 × 8 = (1/2) × 8 = 4$.

38. At 4:00, the minute hand is exactly on the 12 and the hour hand is exactly on the 4. There are $360 ÷ 12 = 30°$ between any two adjacent numbers on a clock, so that’s $4 × 30 = 120°$ between the minute and the hour hands at 4:00. The supplement of 120° is $180 – 120 = 60$ degrees.

39. A 45% discount means we pay $100 – 45 = 55%$, and a 20% discount means we pay $100 – 20 = 80%$. We can now write the following equation, with $p$ representing the original price: $(0.8 × 0.55 × p) – 5 = 50$. Simplifying, we get $0.44 × p = 55$ and then $p = 55 ÷ 0.44 = 125$. The original price must have been $125.

40. The fraction equivalent of 0.125 is 1/8. Dividing by 1/8 is the same as multiplying by 8, because there are 8 eighths in one whole.
Warm-Up 4

41. The first column needs a triangle, and the second row needs a circle. The bottom right corner must be a square because all other rows and columns already contain a square. That means the bottom of the second column must be a triangle, and then a circle is needed right above it. Finally, it is clear that the shaded space must be a triangle (∆).

42. Since the sum of the angles in a triangle is 180° and supplementary angles also add up to 180°, it is always the case that the supplement of an angle in a triangle is equal to the sum of the other two angles. Therefore, the degree measure of our supplement is 7 + 97 = 104 degrees.

43. The decimal equivalent of 7/4 is 1.75, so the desired difference is 3.75 – 1.75 = 2.

44. Two out of 10 equal columns are shaded, so 2/10 = 1/5 of the figure is shaded.

45. The probability that the first student chosen is a girl is 8/20 = 2/5. If a girl is chosen first, then there are only 7 girls left out of the 19 remaining people, so the probability of choosing another girl is 7/19. The probability that both things happen is 2/5 × 7/19 = 14/95.

46. A regular pentagon can be divided into three triangles, as shown, each with an angle sum of 180°. The total interior angle sum of a pentagon is thus 3 × 180 = 540°. In a regular pentagon, this total must be shared equally among the 5 vertices, so each interior angle must be 540° ÷ 5 = 108°.

47. If we let \( n \) represent the number of nickels, \( d \) represent the number of dimes, and \( q \) represent the number of quarters, then we can write the equation \( n + d + q = 37 \). Since we also know that \( n = 4 + d \) and \( q = 2 + n \), it follows that \( q = 6 + d \). By substitution into the first equation, we get \( (4 + d) + d + (6 + d) = 37 \). Simplifying yields \( 10 + 3d = 37 \). Subtracting 10 from both sides and dividing by 3, we get \( d = 9 \), which means that there must be 9 dimes. There are \( 4 + 9 = 13 \) nickels and \( 2 + 13 = 15 \) quarters. To verify these numbers, we add \( 13 + 9 + 15 \), which does indeed equal 37.

48. The 25% active ingredient in the original 12 fl oz of cough syrup measures exactly 1/4 × 12 = 3 oz. If these 3 oz are to be 10% of the final mixture, then the pharmacist needs a total of 30 oz. He will have to add 30 – 12 = 18 fl oz of the flavored syrup.

49. One trillion is the numeral 1 followed by 12 zeros to form a 13-digit numeral. Twenty-one-and-a-half trillion is a 14-digit numeral. In scientific notation, the exponent of the 10 is always one less than the number of digits, so it will need to be 13. The number will look like this: 2.15 × 10¹³.

50. It is a special property of the powers of 2 that the sum of consecutive powers of 2, starting with \( 2^0 = 1 \), is always one less than the next power of 2. Thus, if Safir takes $1 the first day, $2 for the second, and so on, up until \( 2^9 = $512 \) on the tenth day, he will earn \( 2^{10} – 1 = 1024 – 1 = $1023 \). This is $23 more than the lump sum payment of $1000.

Workout 2

51. The full name contains 61 letters. The abbreviated name contains 10 letters, which is 10/61 of the full name. To convert this fraction to a percent, we divide 10 by 61 and multiply the result by 100, which comes to 16.39%, to the nearest hundredth.

52. At 15 mph, John bikes 15/60 = 1/4 of a mile every minute. Dividing 5280 by 4, we find that John must be going 1320 ft every minute.

53. Malcolm will receive $2.00 – $1.64 = $0.36. This can be done with one quarter, one dime and one penny, which is 3 coins.

54. The first, second and fourth rolls can be any number other than 3, and the third roll must be a 3, so the probability that the only 3 occurs on the third roll is \((7/8) \times (7/8) \times (1/8) \times (7/8) = 343/4096\). Simplifying yields \(10 + 3d = 37\). Subtracting 10 from both sides and dividing by 3, we get \(d = 9\), which means that there must be 9 dimes. There are \(4 + 9 = 13\) nickels and \(2 + 13 = 15\) quarters. To verify these numbers, we add \(13 + 9 + 15\), which does indeed equal 37.

55. The area of a rectangle is its length times its width, so we must solve \(15w = 195\). Dividing 195 by 15, we get a width of 13 ft.

56. If we add the 11 students who own cats to the 12 students who own dogs, our sum of 23 double counts the 6 students who own both cats and dogs. Thus, there are actually only 23 – 6 = 17 students who have a cat and/or a dog. Adding to this number the 3 students who have neither a cat nor a dog, we find that there are 17 + 3 = 20 students in Ms. Jones’ pre-algebra class.

57. The length of segment XY is \(8/2 = 4\) cm, which is the height of the trapezoid. We now can compute the area of the trapezoid by multiplying the height of the trapezoid by the average of the lengths of its bases to get \(4 \times (8 + 11)/2 = 4 \times 9.5 = 38\) cm².

58. The combined lengths of \(\overline{AB}\) and \(\overline{BC}\) are the length of \(\overline{AC}\) so we have \((2x + 5) + (6x - 1) = 36\). Combining like terms, we get \(8x + 4 = 36\), which simplifies to \(8x = 32\) and finally \(x = 4\). The length of segment \(AB\) must be \(2 \times 4 \times 5 = 8 + 5 = 13\) cm.

59. Since “replacement” allows the same number to be chosen twice, there are \(5 \times 5 = 25\) ways to pick two numbers from 1 to 5. There are just 2 ways to get a sum of 3, either 1 + 2 or 2 + 1. Thus, the probability is \(2/25\).

60. If we combine the two purchases, the result is exactly double the quantity that we want for each side dish. Using \(p\) for potato salad and \(c\) for coleslaw, we have \(3p + 4c = 10.75\) and \(p + 2c = 4.75\). Together, that’s \(4p + 6c = 15.50\), and half those amounts is \(2p + 3c = 7.75\), so the cost is \$7.75\.

72

MATHCOUNTS 2011-2012
61. A staircase of 1 row requires 1 cube. A staircase of 2 rows takes \(1 + 2 = 3\) cubes. A staircase of 3 rows takes \(1 + 2 + 3 = 6\) cubes. The numbers 1, 3 and 6 are triangular numbers. If this sequence continues, the \(n\)th term equals \(n(n + 1)/2\). Therefore, a staircase of 11 rows requires \(1 + 2 + 3 + \ldots + 11 = 11 \times 12 \div 2 = 66\) cubes.

62. The triangle formed by connecting the centers of the circles is an equilateral triangle with side length equal to the radius of each circle. Since the diameter of each circle is 8 cm, the radius is 4 cm. If we drop a perpendicular from one vertex of the triangle to the opposite side, it will bisect the side. This gives us two 30-60-90 triangles. We know that the length of the hypotenuse of one of these triangles is 4 cm and the length of the short leg is 2 cm. We can use the Pythagorean theorem to find the length of the longer leg, \(h\) (for height), as follows: \(2^2 + h^2 = 4^2 \rightarrow 4 + h^2 = 16 \rightarrow h^2 = 12 \rightarrow h = \sqrt{12} \approx 2\sqrt{3} \approx 3.464\). Using this height and a base of 4 cm, we now find that the area of the equilateral triangle is \((1/2) \times 4 \times 2\sqrt{3} \approx 4\sqrt{3} \approx 6.928\) cm². A nice formula to remember for the area of an equilateral triangle with side length \(s\) is \(A = \frac{s^2 \sqrt{3}}{4}\). The area of the triangle in this problem is then \(4\sqrt{3}/4 = 4\sqrt{3}/4 = \frac{4\sqrt{3}}{4}\) cm².

63. Each term in the geometric sequence is \(-4\) times the previous term, so the next term is \(-4 \times 64 = -256\).

64. The formula for the area of a circle is \(A = \pi r^2\). Since \(9^2 = 81\), the radius of the large circle in the figure must be 9 units. That means the diameter of the large circle is \(2 \times 9 = 18\) units, and the diameter of each smaller circle must be \(18 \div 3 = 6\) units. The circumference of a circle is \(\pi\) times the diameter, so our answer is \(6\pi\).

65. Before solving for \(x\), we might notice that \(12x + 18\) is 6 times as much as \(2x + 3\), so its value is \(6 \times 4 = 24\).

66. The figure shows one way to divide the shaded region into the maximum number of regions, which is 5 regions.

67. There are 32 people represented in the stem-and-leaf plot. The median is the average of the 16th and 17th birth years when the data are listed from least to greatest. In this case, the 16th and the 17th birth years are both 1968, so the median is 1968. The mode is the birth year that occurs most frequently in the data, which is 1974. The positive difference between these two measures is 1974 – 1968 = 6 years.

68. With 6 players on the field at all times, the total player minutes to be filled is \(6 \times 48 = 288\) minutes. Each of the 8 players then can be on the field for \(288 \div 8 = 36\) minutes.

69. The dog leaps 9 – 7 = 2 ft farther than the rabbit each time they leap, so the dog should catch up to the rabbit after \(150 \div 2 = 75\) leaps.

70. The sum of the four smallest numbers must be \(4 \times 80 = 320\), and the sum of the largest four numbers must be \(4 \times 90 = 360\). Since three of the numbers are the same in the two sums, the difference between the largest and the smallest numbers in the set is \(360 – 320 = 40\).

Warm-Up 6

71. There are 6 possible choices for the first person, followed by 5 choices for the second and 4 choices for the third. That would be \(6 \times 5 \times 4 = 120\) different three-member teams if the order in which they were chosen actually mattered. Given that the order doesn’t matter, we have to acknowledge that we have the same group of three people several times over. Suppose, for example, we selected A then B then C. Well, the same three people would be in the group if we had chosen them in the order ACB. Actually, any of the three might have been chosen first, followed by any of the remaining second, and then the last person last. That’s \(3 \times 2 \times 1 = 6\) ways that any group of three might have been chosen. For this reason, we need to divide our 120 by 6, which results in a total of 20 teams. This is known as “6 choose 3,” and the value can be calculated directly as \(6!/[3! \times (6–3)!] = (6 \times 5 \times 4)/(3 \times 2 \times 1) = 20\).

72. The volume of a right circular cylinder is the area of the base times the height. Dividing \(144\pi\) cm³ by the height of 9 cm, we get an area of \(16\pi\) cm² for the base. (Incidentally, we also now know that the radius of the circle is 4 cm.)

73. Seven eight-slice pizzas contain \(7 \times 8 = 56\) slices. After 41 slices are eaten, there are \(56 – 41 = 15\) slices left, which is \(15/56\) of the total amount.

74. Translating the English to math, we get the equation \(v(2x – 3) = 3\). Squaring both sides of the equation, we get \(2x – 3 = 9\), which simplifies to \(2x = 12\) and finally \(x = 6\).

75. Let \(b\) equal the number of marbles Bailey has and \(k\) equal the number of marbles Kaylee has. The statements translate to the following two equations: \(b + 2 = 2(k – 2)\) and \(k + 3 = 3(b – 3)\). Let’s solve the first equation for \(b\) and substitute this into the second equation. Distributing on the right, we get \(b + 2 = 2k – 4\) and then \(b = 2k – 6\). Substituting this into the second equation, we get \(k + 3 = 3(2k – 2 – 3)\), which becomes \(k + 3 = 6k – 27\). Adding 27 to both sides, then subtracting \(k\) from both sides, we get \(30 = 5k\), so \(k = 6\). That means \(b = 2 \times 6 – 6 = 12 – 6 = 6\). It turns out Bailey and Kaylee have the same number of marbles, so the difference is 0 marbles.

76. Although not all the lengths are labeled, we can still find the total perimeter. For example, there are four vertical segments on the left that must add up to 12 ft because that’s the height shown on the right side. Also, there must be two horizontal sums of 7 ft where the sides just inward on the left. We must also include the two sides of length 3 ft at the top of the figure. In all, the perimeter of the figure is \((2 \times 10) + (2 \times 12) + (2 \times 7) + (2 \times 3) + 20 + 24 + 14 + 6 = 64\) ft.
77. If Martha is to average 92 strokes per round on 8 rounds, she will need a total of \(8 \times 92 = 736\) strokes. So far she has made \(5 \times 98 = 490\) strokes. She can make only \(736 - 490 = 246\) more strokes in the next three rounds. That would be an average of \(246 \div 3 = 82\) strokes per round.

78. There are 20 equal intervals in the distance of \(-3 - (-8) = -3 + 8 = 5\) units. Each interval must be \(1/4\) of a unit. There are 10 intervals from \(b\) to \(a\), so the distance is \(10/4 = 2\frac{1}{2}\). Since the greater value is subtracted from the lesser value, however, our answer must be \(-2\frac{1}{2}\). More specifically, \(b = -6\frac{1}{2}\) and \(a = -4\frac{1}{2}\), so \(b - a = -6\frac{1}{2} - (-4\frac{1}{2}) = -6\frac{1}{2} + 4\frac{1}{2} = -2\frac{1}{2}\).

79. A reduction of 20% means you pay 80%, so the new price of the \$80 coat is \(0.8 \times 80 = \$64\). The new price of the \$100 coat, after successive discounts of 30% and 10%, is \(0.7 \times 0.9 \times 100 = \$63\). The positive difference between the prices of the two coats is \$1\) dollar.

80. The region is an isosceles right triangle with base and height of 4 units. The area is \(1/2 \times 4 \times 4 = 8\) units².

**Workout 3**

81. If 2 out of every 3 students attend the Spring Festival, then 1 out of every 3 will not attend. Dividing 1140 by 3, we estimate that 380 students will not attend.

82. We can set up the proportion \(18/x = 60/100\), where \(x\) represents the number of attempted shots. To solve for \(x\), we can multiply \(18 \times 100\) and divide by 60. This gives \(x = 1800/60 = 30\), so Vinnie attempted 30 shots.

83. It will take 8 of the 0.5 cm × 0.5 cm × 0.5 cm cubes to span the 4-cm length, 6 to span the 3-cm width, and 6 to span the 3-cm height. That’s \(8 \times 6 \times 6 = 288\) cubes in all. Alternatively, we could reason that the volume of the cube is \(4 \times 3 \times 3 = 36\) cm³ and it takes \(2 \times 2 \times 2 = 8\) of the smaller cubes to fill each cubic centimeter. Therefore, \(36 \times 8 = 288\) cubes are needed.

84. The total of \$16.20 represents 120% of the cost of dinner and tax. Dividing 16.20 by 1.2, we find that dinner and tax must have cost \$13.50.

85. We know we will need 2 pennies and 40¢. If we consider each coin’s “gram per cent” value, we see a penny is 2.5 g per cent, a nickel is \$5.00 \div 12 = 0.416\) g, and a quarter is \$25 \div 625 = 0.04\) g. Dividing both sides of this equation by π, we get \(7.5 = 1.5\) x. Dividing 7.5 by 1.5, we find that \(x = 5\).

86. We cannot determine the actual area, but we can say for certain that it increases by \(2\) cm². From the figure you can see that the increase in area is the sum of the areas of the \(2 \times \) rectangle and the \(2 \times 2 \) square.

87. The situation calls for proportional reasoning. Segment CD is twice the length of segment BC (\(5340/2670 = 2\)), so segment EF should be twice the length of segment FG. The length of segment EF is \(2 \times 3185 = 6370\) units. A proportion we also could have set up and solved to yield the same result is \(2670/5340 = 1800/60\).

88. Subtracting the 25 messages from the total of 233, we find that the other 8 friends must have received 208 messages. That’s \(208 \div 8 = 26\) messages each.

89. Notice that points A and B have the same x-coordinates. This allows the easy calculation of the length of segment AB as \(10 - 2 = 8\) units. Similarly, points A and C have the same y-coordinates, so the length of segment AC is \(16 - 1 = 15\) units. The interior angle at point A is 90 degrees, so sides AB and AC are the triangle’s base and height, respectively. Thus, the area of the triangle is \((1/2) \times 8 \times 15 = 60\) units².

90. The trapezoids are similar, so the ratios of corresponding sides are the same. Segment AB is 1.5 times the length of segment XW, so segment AD should be 1.5 times the length of segment ZW. This leads to the equation \(1.5(x + 1) = 3x - 6\), which is equivalent to \(0.5x + 1 = 3x - 6\). Subtracting \(0.5x\) from both sides and adding 6 to both sides, we get \(7.5 = 1.5x\). Dividing 7.5 by 1.5, we find that \(x = 5\). Thus, the length of segment AD is \(3 \times 5 = 15\) units.

**Warm-Up 7**

91. To rent a house at the beach for 7 days, it will cost the Sanchez family \(50 + (7 \times 130) = 50 + 910 = \$960\).

92. The formula for the circumference of a circle is \(C = \pi d\), and the formula for the area of a circle is \(A = \pi r^2\). If these two completely different measures are numerically equal, then the following equation is true: \(d = 2r\). Dividing both sides of this equation by \(\pi\), we get \(d = r\). But the diameter is just twice the radius, so we have \(2r = r\), which can only be true if \(r\) is zero or \(2\). Given that there is a circle, the radius must be \(2\) units.

93. The length of the insect viewed under the magnifying glass is \(4 \times 3 = 12\) cm.
94. The sum of the probabilities for each color of the tokens must be 1. So far, we have $1/5 + 1/3 = 3/15 + 5/15 = 8/15$ for the red and the blue tokens. The probability of randomly choosing a green token must be $1 - 8/15 = 7/15$.

95. The least three-digit square is 100, which is $10^2$. The greatest three-digit square is 961, which is $31^2$. Subtracting the 9 square numbers that are less than 100, we get a total of $31 - 9 = 22$ three-digit integers that are squares.

96. First we arrange the 21 numbers from least to greatest as follows: 5, 6, 8, 10, 10, 14, 18, 18, 22, 22, 25, 27, 27, 30, 30, 31, 32, 41, 42, 43, 53. Then we find the middle number in our list, in this case $25$.

97. To get the smallest integer diameters in the ratio 1:2:3:4, we should use diameters of 1, 2, 3 and 4 cm. The smallest total length for the seven beads on the necklace is $1 + 2 + 3 + 4 + 3 + 2 + 1 = 16$ cm.

98. Recall that a number is divisible by 3 if the sum of the digits is divisible by 3. The sum of the three digits 6, 4 and 5 is already a multiple of 3, so D must represent a multiple of 3. The possibilities are 0, 3, 6 and 9. That means the sum of all possible values of D is $0 + 3 + 6 + 9 = 18$.

99. A common denominator for $m$ and $2m$ is $2m$, so we can add the fractions on the left as follows: $1/m + 1/(2m) = 2/(2m) + 1/(2m) = 3/(2m)$. Now we have the equation $3/(2m) = 6$, and we multiply both sides by $2m$ to get $3 = 12m$. Dividing both sides by 12, we get $m = 3/12 = 1/4$.

100. The area of the square is $s^2$. The area of the four isosceles triangles is $4 \times (1/2 \times s \times b) = 2sb$. Subtracting the combined areas of the triangles from the area of the square, we see that the area of the star, in terms of $s$ and $b$, is $s^2 - 2sb$ units$^2$ or $s^2 - 2bs$ units$^2$.

**Warm-Up 8**

101. The perimeter of the first pasture is 2 km, and its area is 25 hectares. We have 8 km of fencing, which will fence a pasture with a perimeter that is four times the perimeter of the first pasture. That means the scale factor (or the ratio of their linear measurements) for these two similar pastures is 1:4. Since area is a two-dimensional measurement, the ratio of their areas will be $1^2 : 4^2 = 1:16$. So, the area of the second pasture is $16 \times 25 = 400$ ha.

102. In the range 101 to 199, there are ten 3s in the tens place and ten 3s in the ones place. In the range 200 to 250, there are still ten 3s in the tens place but only five 3s in the ones place. Angie will need to purchase a total of $3 \times 10 + 5 = 35$ stickers containing the digit 3.

103. Subtracting 2x from both sides of the equation, we get $x + 3 = 7$. Subtracting 3 from both sides, we get $x = 4$.

104. Let $t$ represent the time that Tim drove. Since distance equals rate times time ($d = rt$) and Tim drove at 30 mi/h, we can say that he drove 30t mi. Kim drove at 40 mi/h for 3t hours, so she drove $40 \times 3t = 120t$ mi. Together they drove 150t miles and this is known to be 225 mi. Solving $150t = 225$ for $t$, we get $t = 3/2$ hours. Tim drove for an hour and a half at 30 mi/h, which is a total of $3/2 \times 30 = 45$ mi.

105. Since $A = 1/2 \times b \times h$ for any triangle, it follows that $b = (2 \times A) / h$. We can find the base of $\Delta ABC$ by doubling the area and dividing by the height as follows: $2 \times 243 + 18 = 27 cm$. The ratio of $XZ$ to $BD$ is $4/18 = 2/9$. The length of segment $WY$ is $2/9 \times 27 = 54/9 = 6$ cm.

106. We average the $x$-coordinates and the $y$-coordinates to find the coordinates of the midpoint of the segment. For the given segment, the midpoint has an $x$-coordinate of $(-2 + 3)/2 = 1/2$, and the $y$-coordinate is $(4 + (-3))/2 = 1/2$. So the midpoint has coordinates $(1/2, 1/2) = (0.5, 0.5)$.

107. A Venn diagram is helpful in this situation. Once the 2 is placed in the intersection of all three circles, we can work out the numbers for students participating in at least two of the groups and then those participating in just one. There are 15 students in both band and chorus, which includes the 2 students who are in all three. Thus there are $15 - 2 = 13$ students in only band and chorus. There are 9 students in both orchestra and chorus. Subtracting the 2 students who are in all three, we get $9 - 2 = 7$ students who are in orchestra and chorus only. Next, there are 4 students in band and orchestra, 2 of whom also participate in chorus. That means there are $4 - 2 = 2$ students in only band and orchestra. We can now see that there are $21 - (7 + 2 + 2) = 10$ students who are in orchestra only.

108. The range of the data is the maximum value minus the minimum value, which is $16 - 4 = 12$ in this set. The mean is the sum of the data divided by the number of items in the data set, which is $(4 + 5 + 7 + 7 + 8 + 8 + 8 + 9 + 16)/9 = 72/9 = 8$. The positive difference between the range and the mean of the data is $12 - 8 = 4$.

109. Two adjacent sides of the rectangle and a diagonal form a right triangle, which allows us to use the Pythagorean Theorem to find the length of the diagonal. The length of the diagonal is equal to $\sqrt{18^2 + 24^2} = \sqrt{(324 + 576) = \sqrt{900) = 30}$ m. Some students will notice that $18$ is $3 \times 6$ and $24$ is $4 \times 6$, so the diagonal is $9 \times 6 = 30$ m. This is a multiple of the $3$-$4$-$5$ triangle, which satisfies the Pythagorean Theorem. The sum of the lengths of the diagonals is thus $30 + 30 = 60$ m.

110. To find the average miles per hour, we divide the miles by the hours ($r = d/t$). In this case, if we convert $2.25$ to $9/4$, then the division problem $117 \div 2.25$ becomes $117 \div 9/4$. This has the same answer as the multiplication problem $117 \times 4/9$, which is $13 \times 4 = 52$ mi/h.
111. Since the two cones will be similar, the ratios of height to radius for the two cones will be equal, so we can solve the proportion \(18/8 = x/2\) for \(x\). Looking at the denominators, we notice that 2 is 1/4 of 8, so \(x\) is 1/4 of 18. Dividing 18 by 4, we get a height of 4.5 in.

112. To find the GCF of two numbers, it helps to look at the prime factorizations: 72 = \(2^3 \times 3^2\) and 48 = \(2^4 \times 3\). The two numbers have three factors of 2 and one factor of 3 in common, so the GCF is \(2^3 \times 3 = 24\). Similarly, we have 108 = \(2^2 \times 3^3\) and 144 = \(2^4 \times 3^2\), so their GCF is \(2^2 \times 3^2 = 36\). For the LCM, we find the product of the greatest power of each prime factor, so for 24 and 36, we need three factors of 2 and two factors of 3, which is \(2^3 \times 3^2 = 72\). Indeed, \(3 \times 24 = 2 \times 36 = 72\).

113. Using the Fundamental Counting Principle, we see there are \(2 \times 24 \times 24 \times 24 = 27,648\) possible arrangements for station call letters.

114. The hare will travel 60 m in one hour. The tortoise will travel 60 cm in one minute, which is equal to \(60 \times 60 = 3600\) cm in one hour. Since 100 cm = 1 m, 3600 cm = 36 m. In one hour the hare will travel 60 – 36 = 24 m farther than the tortoise.

115. There are \(24 \times 60 = 1440\) minutes in every day. If there is a net gain of 150 people every minute, there must be a net gain each day of \(150 \times 1440 = 216,000 = 2.16 \times 10^5\) people.

116. An increase of 75% means that the new length of the rectangle is 1.75 times the original length. Similarly, an increase of 25% in the width means the new width is 1.25 times the original width. The new area is \(1.75 \times 1.25 = 2.1875 = 218.75\%\) of the original area. This represents an increase of \(218.75 - 100 = 118.75\%\).

117. At the start, the 50 mL of 50% acid solution is 25 mL acid and 25 mL water. Each time 10 mL of the 10% acid solution is added, we get 1 mL acid and 9 mL water. Let \(n\) represent the number of times we add 10 mL of the 10% acid solution to the original solution. The amount of acid is \(100 + 4n = 25 + 9n\). Solving for \(n\), we get \(100 + 4n = 25 + 9n\), then \(75 = 5n\) and finally \(n = 15\). Recall that \(n\) represents the number of times we added 10 mL to the original solution, so we will have added \(15 \times 10 = 150\) mL to the original 50 mL. That gives us a total of 200 mL of 20% acid solution.

118. We can rewrite \(42 ÷ 3.5 = 42 ÷ (7/2)\), which is \(42 \times (2/7) = 12\). This means that each inch on the map represents 12 miles. The distance of 118 mi. farther than the tortoise.

119. We already know the height of the first bounce, so we need to multiply 5 feet by 0.8 five times to get to the sixth bounce: \(5 \times 0.8^5 = 1.6\) ft. To the nearest tenth, that’s 1.6 ft.

120. A circle with a diameter of 8 \(\frac{1}{2}\) in has a radius of \(4 \frac{1}{2}\) in. The area of the circle is \(\pi \times 4.25^2 = 18.0625\pi \approx 56.75\) in\(^2\). The area of the full sheet of paper is \(8.5 \times 11 = 93.5\) in\(^2\). The leftover part of the sheet of paper is about \(93.5 – 56.75 = 36.75\) in\(^2\). This is about \(\frac{36.75}{93.5} \times 100 = 39.3\%\) of the full sheet of paper.

Warm-Up 9

121. Superimposing the grids, as shown, we see that the letter H is formed.

122. If the pole is four parts above ground and one part below ground, that’s five parts in all. Each part must be \(30 ÷ 5 = 6\) ft. The top of the pole must be \(4 \times 6 = 24\) ft above ground.

123. Given the first two conditions, the seven possible numbers are 930, 840, 750, 660, 570, 480 and 390. To get the difference of 540 by swapping the tens and hundreds digits, the numbers must be 930 and 390, with the original number being 390.

124. For each of the six numbers that could land up on the red die, there are six possibilities on the blue die, followed by another six possibilities on the green die. Using the Fundamental Counting Principle, we get \(6 \times 6 \times 6 = 216\) possible outcomes.

125. If the vertex angle of the isosceles triangle is 50°, then the other \(180 - 50 = 130°\) of the triangle must be divided equally between the two base angles. That means angles F and G are each 65°. By properties of corresponding angles, we know that \(\angle BDE\) and \(\angle CED\) also measure 65°. Angle \(\angle DEG\) is supplementary to \(\angle CED\), so its measure is \(180 - 65 = 115°\). This angle corresponds to \(\angle BCE\), so the measure of \(\angle BCE\) is 115°.

126. If we divide the larger square into a 3-by-3 grid, as shown in the figure, the shaded triangles occupy 4 out of the 9 square units. That represents, to the nearest whole number, \(4 ÷ 9 \times 100 = 44.4\% = 44\%\).

127. The probability that Christoph will get an A on the first test is 0.25 = 1/4. The probability that he will earn an A on the following test is also 1/4. Thus, the probability that Christoph will get an A on his next two tests is \(1/4 \times 1/4 = 1/16\).

128. Given that the area of right \(\triangle CDE\) is 30 units\(^2\) and one leg has length 5 units, we can calculate that the length of the other leg, segment CE, must be \(2 \times 30 ÷ 5 = 12\) units. This means segment BC must be \(3 \times 2 \times 8\) units. We now have the two legs of right \(\triangle ABC\) so, by the Pythagorean Theorem, the length of segment AB is \(\sqrt{(5^2 + 8^2)} = \sqrt{(25 + 64)} = \sqrt{89}\) units.
129. The number of milk balls Mandy started with was $5 + 3 \times (1 + 3 + 5 + 7 + 9 + 11 + 13 + 15)$. From the eight consecutive odd numbers, we notice four sums of 16. So, $5 + 3 \times (1 + 15) + (3 + 13) + (5 + 11) + (7 + 9) = 8 + 4 + 16 = 8 + 64 = 72$ was the number of milk balls Mandy started with.

130. Note that we need not worry about the entire line, just $P$. When the point $P(3, 2)$ is reflected over the $y$-axis, the $x$-coordinate changes sign, so the image is $P'(-3, 2)$. Now, when this point is translated up 4 units, the $y$-coordinate is 4 more than it was, so the final image is $P'(–3, 6)$.

Warm-Up 10

131. We can get 24 from each pair as follows: $6 \times 4, 16 + 8$ and $36 – 12$.

132. We can ignore Grandpa’s three quarters and just think about $18 – 3 = 15$ dimes and nickels. Because the coins are 2 parts dimes and 1 part nickels, there are 3 parts in all and each part must be $15 + 3 = 5$ coins. That means there are $2 \times 5 = 10$ dimes and 5 nickels. The total value of these coins is $(10 \times 0.10) + (5 \times 0.05) = 1.00 + 0.25 = \$1.25$.

133. The vertical band where the two squares overlap must be $2 \times 12 – 20 = 24 – 20 = 4$ units wide. That means the rectangular region on each side is 8 units by 12 units. The triangle must have an area of $1/2 \times 8 \times 12 = 48$ units$^2$.

134. The sets of distinct, positive integers with a product of 84 are $\{2, 42\}, \{2, 6, 7\}, \{2, 3, 14\}, \{3, 28\}, \{3, 4, 7\}, \{4, 21\}, \{6, 14\}$ and $\{7, 12\}$. These sets have the sums 44, 19, 11, 25, 20 and 19, respectively. Thus, the least possible sum is 14.

135. It takes the artist $240 \div 20 = 12$ minutes to draw each picture. If she draws 3 times as fast, it will take $1/3$ of the time per picture, or 4 minutes per picture. Six hours is $6 \times 60 = 360$ minutes. In 6 hours she can draw $360 \div 4 = 90$ pictures.

136. If we consider 12:00 to be 0° and move clockwise, then the number 3 is at 90° and the number 4 is at 120°. At 3:30, the hour hand will be halfway between these two measures, which is 105°. The minute hand will be on the 6, which is at 180°. The measure of the angle between the minute and hour hands is $180 – 105 = 75°$.

137. Each square face of a cube is $1 \times 1 = 1$ cm$^2$. Therefore, we just need to count the square faces of the cubes to get the surface area. There are 8 squares on the left, 8 squares on the right, 10 squares in front, 10 squares in back, 7 squares on top and 7 squares on the bottom. That’s $2 \times (8 + 10 + 7) = 2 \times 25 = 50$ squares in all, giving us a total surface area of 50 cm$^2$.

138. The slope of this line is $(43 – 3)/(17 – 1) = 40/16 = 5/2$. If we start at $P(1, 3)$, we will land on another lattice point (a point with integer coordinates) every time we move up 5 units and right 2 units. If we make this move 8 times, we will land on $Q(17, 43)$. So there must be 8 possible lattice points between points $P$ and $Q$.

139. We can create an organized list such as the one shown here. There are $8 + 6 + 5 + 3 + 2 = 24$ sets of three distinct positive integers shown here.

140. For each of the 7 doors a person can enter through, there are 6 possible doors he can exit through. That’s $7 \times 6 = 42$ ways.

Workout 5

141. Although we speak in terms of percents on occasion, we compute with either the fraction or the decimal equivalent. Thus, 40% of 80 can be expressed as $2/5 \times 80 = 2 \times 16 = 32$ or $0.4 \times 80 = 32$. Similarly, 32% of 75 is $8/25 \times 75 = 8 \times 3 = 24$ or $0.32 \times 75 = 24$. The difference between these two numbers is $32 – 24 = 8$, and 75% of 8 is $3/4 \times 8 = 6$ or $0.75 \times 8 = 6$.

142. There are 8 possible digits for the first digit of the area code and then 10 possible digits for the next two digits of the area code. That’s $8 \times 10 \times 10 = 800$ possible area codes, but we have to subtract the 4 toll-free codes, so there are actually 796 possible area codes. For the local code, there are $8 \times 10^2 = 8,000,000$ possibilities. Multiplying 796 by 8,000,000, we see that the total number of possible ten-digit phone numbers is $6,368,000,000 = 6.368 \times 10^9$.

143. The hexagon can be divided into six equilateral triangles with side lengths equal to the radius of the circle. To find the area of the shaded region, we will subtract the area of the six triangles from the area of the circle. Using the formula for the area of an equilateral triangle, the area of the hexagon is $6 \times (s^2 \sqrt{3})/4 = 6 \times (2^2 \sqrt{3})/4 = 6 \times 3$ in$^2$. The area of the circle is $\pi \times 2^2 = 4\pi$ in$^2$. Subtracting 6\sqrt{3}$ from $4\pi$, we find that the area of the shaded region, to the nearest hundredth, is 2.17 in$^2$.

144. Let $x$ be the cost of the first bike and $y$ be the cost of the second bike. The profit on the first bike is 0.3$x$ and the profit on the second bike is 0.5$y$. The sum of the profits is equal to a 45% profit for both bikes, so we can write $0.3x + 0.5y = 0.45(x + y)$. Distributing on the right, we get $0.3x + 0.5y = 0.45x + 0.45y$, which can be simplified further to $0.05y = 0.15x$. We are looking for the ratio of the cost of the first bike to the cost of the second bike, so we solve for $x/y$ and get $0.05/0.15 = 1/3$.

145. The best we can do in the hundreds place is to pick two numbers that differ by 1. There are four sets of these $\{8, 7\}, \{7, 6\}, \{4, 3\}$ and $\{3, 2\}$. Then we want the least two-digit number with the greater hundreds digit and the greatest two-digit number with the lesser hundreds digit. For example, $823 – 764 = 59$ is pretty good, but we can do better with $723 – 684$ or $426 – 387$, which each yield 39.
146. By properties of 30-60-90 triangles the length of the longer leg is always $\sqrt{3}$ times the length of the shorter leg. In this case $CD = 40$ mm, so $AD = 40\sqrt{3}$ mm. Using the square root button on a calculator, we find that the area of rectangle $ABCD$ is $40 \times 40\sqrt{3} = 2771.3$ mm$^2$.

147. Let $q$ represent a quitch and let $g$ represent at gritch. We have $q + 2g = 20$ and $2q + g = 25$. Solving the first equation for $q$ yields $q = 20 - 2g$. We can substitute this expression for $q$ in the second equation to obtain the following equation: $2(20 - 2g) + g = 25$. Distributing the 2 and combining like terms, we get $40 - 3g = 25$. Subtracting 40 from each side, we get $-3g = -15$. Finally, dividing each side by $-3$, we have that each gritch is worth $g = 5$ points.

Alternatively if we combine the two pieces of information provided in the problem, we get 3 quitches and 3 gritches that are worth 45 points. If three of each are worth 45 points, then one of each should be worth $45 ÷ 3 = 15$ points. Comparing this to the first statement where 1 quitch and 2 gritches are worth 20 points, we can see that the extra gritch must be worth 5 points.

148. Let $rt$ be the travel time, in hours, at 30 mi/h, and let $d$ be the distance traveled, in miles. Since distance equals rate times time ($d = rt$), we have $d = 30t$. Since the second trip takes a total of 26 minutes less than the first trip, we can write $d = 45(t - 26/60)$ for the second trip. The distance is the same going to work and coming home, so we can equate the two expressions, giving us $30t = 45(t - 26/60)$. Distributing the 2 and combining like terms, we get $40 - 3g = 25$. Subtracting 40 from each side, we get $-3g = -15$. Finally, dividing each side by $-3$, we have that each gritch is worth $g = 5$ points.

Warm-Up 11

151. Since there are five columns on this particular table, we are looking for possible values of $n$ that have exactly five multiples that are less than 36. That excludes numbers greater than 7. We know $n$ is not 1 since every number on the table is a multiple of 1. If $n$ is 2, 3, 4 or 5 there is more than one multiple of $n$ in at least one column. If $n$ is 6 or 7, however, we see that there is exactly one multiple of $n$ in each column. So, Mara might have colored either the multiples of 6 or the multiples of 7, and the sum of all possible values of $n$ is $6 + 7 = 13$.

152. There are five possible positive differences of 1 when two dice are rolled, but each of them can happen in two ways. For example, if we roll a 6 on a red die and a 5 on a blue die, the positive difference is $6 - 5 = 1$, but we can also roll a 5 on the red die and a 6 on the blue die. The same goes for 5 – 4, 4 – 3, 3 – 2 and 2 – 1. In all, there are 10 ways to get a difference of 1 out of 36 possible outcomes when a pair of dice is rolled, so the probability is $10/36 = \frac{10}{36}$.

153. The slope of a perpendicular line is the negative reciprocal of the slope of the line to which it is perpendicular. The slope of the line passing through $A(–8.1, 4.9)$ and $B(–7.6, 2.9)$ is $(2.9 - 4.9)/(–7.6 – (–8.1)) = –2/0.5 = –4$. The slope of the perpendicular line is $\frac{1}{4}$.

154. The digit 5 plays a special role in this problem. First, for the value of the units place of the answer, since an even number times 5 produces a zero, which we don’t have, and an odd number times 5 produces another 5, which we don’t have, the 5 cannot be in the units place of any of the three numbers. Also, since we are only using the digits 1 through 6, our product will certainly be less than 500. This means the 5 can only be in the tens place of either the first factor or the product. With some trial and error, we eventually find $54 \times 3 = 162$, as shown.

155. There are 16 Round 1 games worth 1 point each, for a total of 16 points. There are 8 Round 2 games worth 2 points each, for a total of $2 \times 8 = 16$ points. There are 4 Round 3 games worth 4 points each, for a total of $4 \times 4 = 16$ points. There are 2 Round 4 games worth 8 points each, for a total of $2 \times 8 = 16$ points. Finally, there is 1 Round 5 game worth 16 points. That’s $5 \times 16 = 80$ points, in all.

156. Since the circumference of the cone is triple the circumference of the cylinder, its radius must also be 3 times the radius of the cylinder. Let $r$ be the radius of the cylinder. That means the radius of the base of the cone is $3r$. The volume of a cylinder equals the product of the area of its base and its height. So the area of this cylinder is $3\pi r^2$. The volume of a cone is $1/3$ of the product of the area of its base and its height. For the cone the volume is $\{1/3\}r^2h = (1/3)\pi 3^2h = 3\pi r^2$. The ratio of the volume of the cylinder to the volume of the cone is then $\pi r^2/3\pi r^2 = 1/3$.

157. If we keep track of the number of ways to get to each intersection, we can simply add the numbers from the higher intersections to get the number for a new intersection, as shown. We get something that starts out like Pascal’s triangle, but loses its symmetry where the line segment is missing. As the figure shows, there are 17 paths from A to Z.

158. When a difference is reversed, we get the opposite result. We will start this solution by reversing the $x – y$ in the denominator. We now have that $(x - y)/(y - z) = 2$. Multiplying each side of the equation by the quantity $y – z$ yields $x - y = 2(y - z)$. After distributing and adding $y$ to each side, we have $x = 3y – 2z$. Now let’s subtract $z$ from each side to get $x - z = 3y - 3z$. When we factor out a 3 on the right side, the result is $x - z = 3(y - z)$. Finally, dividing each side of the equation by the quantity $y – z$ we see that $(x - z)/(y - z) = 3$.
159. To get the area of the shaded regions, we will subtract the area of the small circle from the area of the big circle, and we will add the area of the square. The diagonal of the square is 8 units, so based on properties of 45-45-90 triangles, its side length must be 4√2 units. The area of the shaded square is (4√2)² = 32 in². Next we find the area of the large outer circle, with radius of length 8 in. So, the area of the outer circle is π8² = 64π in². The area of the inner circle with radius of length 4 in is π4² = 16π in². So, the total area of the shaded regions is 64π – 16π + 32 = 48π + 32 in².

160. Each of the 10 players plays 9 other players, which is 10 × 9 ÷ 2 = 45 matches. This means there must be 45 wins and 45 losses. If 6 people have 7 wins, that’s 42 wins. It is not possible for the remaining 4 people to account for the corresponding 42 losses, since they only play a total of 4 × 9 = 36 matches, and 6 of those are against each other. It is, however, possible for 5 players to have 7 or more wins.

Warm-Up 12

161. Let’s say the width of the pool is x ft. Then the length of the pool is 1.5x ft. We know the area of the pool is 216 ft², so we have x × 1.5x = 216 → 1.5x² = 216. Dividing both sides by 1.5, we get x² = 144 and we see that x = 12 ft. The pool is 12 ft by 1.5 × 12 = 18 ft, so the outer edge of the deck forms a rectangle that is 5 + 12 + 5 = 22 ft wide and 5 + 18 + 5 = 28 ft long. So, the outside perimeter of the deck is 22 + 28 + 22 + 28 = 100 ft.

162. We need to find the least common multiple (LCM) of 25 and 45. Now, 25 = 5 × 5 and 45 = 5 × 3 × 3, so the LCM of the two numbers must be 5 × 5 × 3 × 3 = 225. Dividing 225 by 60, we get 3 remainder 45, so it will be 3:45 pm when the subway and the train next arrive at the station together.

163. Since Tom digs twice as fast as Dick, in the 6 hours, Tom would dig 2 of the holes and Dick would dig the other hole. This means Tom can dig 1 hole every 3 hours. So, to dig 12 holes on his own, it would take Tom 3 × 12 = 36 hours.

164. From 300 to 363, there are 64 integers with the digit 3 in the hundreds place. From 30 to 39, 130 to 139, 230 to 239 and 330 to 339 there is 1 hole every 3 hours. So, to dig 12 holes on his own, it would take Tom 3 × 12 = 36 hours.

165. It is possible to make squares of five different sizes on a 16-pin geoboard. As shown, it is possible to form a 1 × 1 square, a 2 × 2 square, a 3 × 3 square, a 4 × 4 square and a 5 × 5 square.

166. If 36 is the median, we know the numbers are arranged a₁, a₂, a₃, a₄, a₅. We also know that (3/2) a₁ = 36, so a₁ = (2/3)36 and a₁ = 24. That also means that a₄ = (2/3)24 = 16. We also can determine that a₅ = (3/2)36 = 54 and a₂ = (3/2)54 = 81. Thus the five numbers are 16, 24, 36, 54 and 81. The sum is 211 and the mean is 211 ÷ 5 = 42.2.

167. Let m be Madison’s age now and h be Harper’s age now. Then the information given translates to the following: m + 3h = 47 and m + 2 = 2(h + 2). Solving the first equation for m, we get m = 47 – 3h. We can now substitute this into the second equation, which becomes (47 – 3h) + 2 = 2(h + 2), which simplifies to 49 – 3h = 2h + 4. Solving for h, we get 45 = 5h, then h = 9. Harper must be 9 years old now.

168. Andie and Deanne both bought 2 oldies CDs and 2 current CDs. The difference in their purchases was in the fifth CD they each bought. The difference in the prices of the fifth CD Andie and Deanne each purchased accounts for the difference in the total amount they paid, which is 82 – 78 = 4 dollars.

169. The only one-digit number that meets these criteria is 5. For two-digit numbers we know that 1 and 4 have a sum of 5, as do 2 and 3. Thus, there are 4 two-digit numbers (14, 41, 23, 32) that meet these criteria. For three-digit numbers we can have any permutation of a 3 and two 1s or a 1 and two 2s. That’s a total of 6 three-digit numbers (311, 131, 113, 221, 212, 122). For four-digit numbers we can have any permutation of a 2 and three 1s. There are 4 four-digit numbers (2111, 1211, 1121, 1112). Finally, 1111 is the only five-digit number whose digits have a sum of 5. So, the total number of possible integers with a sum of digits equal to 5 and no digit of zero is 1 + 4 + 6 + 4 + 1 = 16 integers.

170. The length of side AB must be 15 units, since 9-12-15 is a multiple of the 3-4-5 Pythagorean triple. If we draw segment DE parallel to segment AC, then ΔBED = ΔBCA, with the lengths of sides of ΔBED equal to 2/3 the lengths of the corresponding sides on ΔBCA. This means ED = 8 units. Now we can use the Pythagorean Theorem on right ΔDEC to find the length of segment CD as follows: x = √(3² + 8²) = √(9 + 64) = √73 units.

Workout 6

171. We know that the number of pennies Mollie has is a multiple of 3. We also know that when the number of pennies is divided by 5 the remainder is 1. Since multiples of 5 have a units digit of either 0 or 5, Mollie’s total quantity of pennies must have a units digit of either 0 + 3 = 3 or 5 + 3 = 8. Let’s consider the following multiples of 3 that end in 3 or 8: 18, 33, 48, 63, 78 and 93. Of these, only 78 is one more than a multiple of 5. Thus the only 5-digit number whose digits have a sum of 5 is 16785.

172. If we draw radii to points A and B from the center of the circle, as shown, then we have an equilateral triangle. The measure of angle BAD is equal to the measure of its central angle, ∠ABD, which equals 60 degrees. The measure of inscribed angle ∠ACB is equal to half the measure of AB. Therefore, the measure of ∠ACB is 60 ÷ 2 = 30 degrees.
173. When an item is marked 20% off, the customer has to pay 100 – 20 = 80% of the cost. Similarly, when a 5% tax is added, the customer pays a total of 105% of the price. In this case, Cyndi pays 105% of 80% of the cost, or \(1.05 \times 0.8 = 0.84\) or 84% of the original price. Dividing $15.54 by 0.84, we calculate that the original price of the shirt must have been $18.50.

174. The length of the rectangle is equal to the sum of the diameters of circle \(P\), circle \(O\) and circle \(Q\). The radius of circle \(O\) is 4 in, which means its diameter is \(2 \times 4 = 8\) in. The diameters of circle \(P\) and circle \(Q\) are each equal to the radius of circle \(O\), which is 4 in. Thus, the length of the rectangle is \(4 + 8 + 4 = 16\) in. The area of the rectangle is \(16 \times 8 = 128\) in\(^2\). From this, we will subtract the areas of the circles. We have two circles, each with an area of \(4\pi\) in\(^2\), and one circle with an area of \(16\pi\) in\(^2\). That's a total area of \(8\pi + 16\pi = 24\pi\) in\(^2\) for the circles. So, the shaded area is about \(128 – 24\pi \approx 52.6\) in\(^2\).

175. Since all primes greater than 2 are odd and the sum of two odds is even, one of our two primes must be 2. Also, since \(P - Q\) must be positive, the prime \(Q\) and not \(P\) has to be 2. Now, the difference between \(P + 2\) and \(P - 2\) is 4. There are infinitely many pairs of primes that are 4 apart, but we also need \(P\) to be prime. This means we need three consecutive odd primes, \(P - 2\), \(P\) and \(P + 2\). The only time this happens is with the primes 3, 5 and 7. After that, one of the three consecutive odds is always divisible by 3. Thus, the only possible value of \(S\) is \(2 + 5 + 7 + 3 = 17\).

176. Since \(r \times t = d\), it follows that \(t = d/r\). Bob ran 80 yd while Rob ran 80v2 yd, since the length of the hypotenuse of a square is v2 times the length of the side. We know Bob’s rate, and we know their times were equal. For Bob we have \(t = 80/8\). For Rob we have \(t = 80v2/r\). Note that since the units for both fractions are identical, they are no longer related. Setting the two expressions equal to each other, we have \(80/8 = 80v2/r\). Cross multiplying and solving for \(r\), we get \(80r = 8(80v2) \rightarrow r = 8v2 = 11.3\) mi/h.

177. The sum of the four known numbers of the set is \(5 + 9 + 10 + 12 = 36\). The median has to be either 9 or 10. To get a mean of 9, we would need the numbers to have a sum of \(5 \times 9 = 45\), so \(n\) would have to be 45 – 36 = 9. To get a mean of 10, the sum would need to be \(5 \times 10 = 50\), so \(n\) would have to be 50 – 36 = 14. Those are the 2 integers for which the mean and the median of the set are equal.

178. According to the table, a total of \(68 + 12 + 24 + 44 + 52 = 200\) students voted. The number of students who did not vote for Grizzly Bear is 200 – 68 = 132 students. Thus, the probability that a randomly selected student did not vote in favor of Grizzly Bear is 132/200 = 33/50.

179. Consider the prime factorization of 96 = \(2^5 \times 3\). So, any factor of 96 can contain at most five factors of 2. In other words, there could be 0, 1, 2, 3, 4 or 5 factors of 2 (six possibilities). Similarly, there are two possibilities for the number of the factors of 3. They are 0, 1. Using the Fundamental Counting Principle, we have that there are \(6 \times 2 = 12\) positive integer factors. In general, we can add 1 to each exponent in the prime factorization and multiply the resulting values to get the number of positive integer factors of a number.

180. One way to find the area of this quadrilateral is to build a rectangle around it, as shown, and subtract the areas of the four right triangles that are not part of the shaded region. The rectangle would have an area of \(11 \times 14 = 154\) units\(^2\). The areas of the four right triangles are \((1/2) \times 5 \times 10 = 25\) units\(^2\), \((1/2) \times 4 \times 8 = 16\) units\(^2\), \((1/2) \times 3 \times 6 = 9\) units\(^2\) and \((1/2) \times 6 \times 8 = 24\) units\(^2\). Thus, the area of the quadrilateral is \(154 – (25 + 16 + 9 + 24) = 154 – 74 = 80\) units\(^2\).

**Warm-Up 13**

181. The table indicates the number of rectangles of varying dimensions contained in the figure. There are a total of 36 rectangles and 14 of them are squares. So the probability that a rectangle chosen at random is a square is \(14/36 = 7/18\).

<table>
<thead>
<tr>
<th>Dimensions</th>
<th>1 × 1</th>
<th>1 × 2</th>
<th>2 × 1</th>
<th>2 × 2</th>
<th>1 × 3</th>
<th>3 × 1</th>
<th>3 × 2</th>
<th>2 × 3</th>
<th>3 × 2</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quantity</td>
<td>9</td>
<td>6</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>36</td>
</tr>
</tbody>
</table>

182. To determine the area of the triangle, we need the length of its base, which is unknown. If we consider the height of 5 cm and half the base, we have a 5-12-13 right triangle. If half the base is 12 cm, the full base must be 24 cm. The area of the triangle is equal to \((1/2) \times 24 \times 5 = 60\) cm\(^2\).

183. The even numbers between 0 and 10 are 2, 4, 6, and 8, so we are looking for the sum \((1/2) + (1/4) + (1/6) + (1/8)\). The least common denominator is 24, so we get \((12 + 6 + 4 + 3)/24 = 25/24\).

184. We know that for a rectangle \(A = lw\). That means \(2a^2 – ab – b^2 = (a – b)(2a + b)\). When we factor the area, we get \(2a^2 – ab – b^2 = (a – b)(2a + b)\). Therefore, if the width is \(a - b\), the length must be \(2a + b\).

185. Since \(d = r \times t\), we see that Mike first travels \(3 \times 20 = 60\) mi and then travels another \(2 \times 30 = 60\) mi, for a total distance of 120 mi. He travels this distance in 5 hours, so Mike’s average speed is \(120 \div 5 = 24\) mi/h.

186. There are 9 choices for the first digit, since zero cannot be first. Then there are 9 choices for the second digit, 8 choices for the third and 7 choices for the fourth. Using the Fundamental Counting Principle, we see that there are \(9 \times 9 \times 8 \times 7 = 4536\) different four-digit integers.

187. A right circular cylinder is made up of two circles, each with an area of \(\pi r^2\), and a rectangle with an area of \(2\pi rh\) (circumference of a circular base \(\times\) height of the cylinder). Since the height of our cylinder is twice the diameter of the circle, it must be four times the radius. This means we can substitute \(4r\) for \(h\) in the formula for the surface area. We proceed as follows: \(160\pi = 2\pi r^2 + 2\pi rh \rightarrow 160\pi = 2\pi r^2 + 2\pi(4r) \rightarrow 160\pi = 2\pi r^2 + 8\pi r^2 \rightarrow 160\pi = 10\pi r^2\). Dividing each side by 10\(\pi\), we have \(16 = r^2\), so \(r = 4\) cm and \(h = 4 \times 4 = 16\) cm. The volume of the cylinder is the area of the base times the height, so we have \(V = \pi r^2 h = \pi \times 4^2 \times 16 = 256\pi\) cm\(^3\).
188. Let $b$ represent the number of animals in Ben’s collection and $j$ represent the number of animals in Jerry’s collection. Since Jerry has 10 more animals than Ben, we have that $j = b + 10$. We also know that the total number of legs in both collections combined is 220 legs. So, $2b + 4j = 220$. We can substitute the quantity $b + 10$ for $j$ in this equation to get $2b + 4(b + 10) = 220$. Simplifying and solving for $b$, the result is $2b + 4b + 40 = 220 \rightarrow 6b + 40 = 220 \rightarrow b = 180 \rightarrow b = 30$. Therefore, Jerry’s collection contains 30 + 10 = 40 toy animals.

189. There are 10! = 10 × 9 × 8 × 7 × 6 × 5 × 4 × 3 × 2 × 1 = 3,628,800 ways to arrange ten distinct items in a row. In this problem, however, the blue and green marbles are not considered distinct, so we divide this product by the 5! = 5 × 4 × 3 × 2 × 1 = 120 ways that the blue marbles might be arranged and again by the 5! = 5 × 4 × 3 × 2 × 1 = 120 ways that the green marbles might be arranged. The result is $3,628,800/[120][120] = 252$ ways that the blue and green marbles can be arranged. Of these 252 arrangements, one has alternating colors starting with blue, and the other has alternating colors starting with green. The probability that the marbles are arranged with alternating colors is $2/252 = 1/126$.

190. When the edge length of a cube is doubled, the volume is 8 times as great because the doubling happens in all three dimensions. That is, if $V = s^3$ and the side length increases to 2s, then we have $(2s)^3 = 2^3 \times s^3 = 8s^3 = 8V$. The edge length must be increased by 100% to achieve a cube that is 8 times the original cube’s volume.

**Warm-Up 14**

191. The prime factorization of 2940 is $2 \times 2 \times 3 \times 5 \times 7 \times 7$. The 4 perfect square factors of 2940 are 1, $2 \times 2$, $7 \times 7$ and $2 \times 2 \times 7 \times 7$, also known as 1, 4, 49 and 196.

192. If Michael hits the same region with all three darts, the possible scores are 1 + 1 + 1 = 3, 5 + 5 + 5 = 15 and 10 + 10 + 10 = 30. If he hits one region twice and another region once, the possible scores are 1 + 1 + 5 = 7, 1 + 1 + 10 = 12, 5 + 5 + 1 = 11, 5 + 5 + 10 = 20, 10 + 10 + 1 = 21 and 10 + 10 + 5 = 25. Finally, there is only one score if the three darts hit the three different regions: 1 + 5 + 10 = 16. That’s 10 different total scores Michael could earn.

193. We can see that $2^{14} = 2^{(2 \times 7)} = (2^2)^7 = 4^7$. We also can see that $(1/4)^3 = (4^{-1})^3 = 4^{-3}$. If $4^3 = 4^x$, then $9 = -x$, and $x = -9$.

194. The diameter is a linear dimension, whereas volume exists in three dimensions. Therefore, when the diameter of the balloon is doubled, the volume will be $2 \times 2 \times 2 = 8$ times as great. The original volume was 5 cubic meters of air, so the new volume is $5 \times 8 = 40$ m$^3$.

195. Triangles BPM and APN are 30-60-90 right triangles, so the lengths of their respective hypotenuses are twice the lengths of their shorter legs. If $BM = x$, then $BP = 2x$. Similarly, if $AN = y$, then $AP = 2y$. We are told that $AB = 12$ cm, which means that $2x + 2y = 12$. This can be reduced to $x + y = 6$. The value of $CM + CN = (12 – x) + (12 – y)$, which can be rewritten as $24 – (x + y)$. Substituting 6 for $x + y$, we get that the value of $CM + CN = 24 – 6 = 18$ cm, which does not depend on the location of point P.

196. Using properties of right triangles, we can determine the area of the square labeled J. If we let $r$ represent the side length of the square labeled J, then its area is $r^2$. From the Pythagorean Theorem, we know that $r^2 + 2^2 = 7^2$.

197. Kathy ate 1/8 of the jelly beans, leaving $1 – (1/8) = 7/8$ of the jelly beans in the jar. Sue ate $(1/5) \times (7/8) = 7/40$ of the jelly beans. Together Kathy and Sue ate $(1/8) + (7/40) = (5/40) + (7/40) = 12/40$ of the jelly beans. Pat ate twice this amount, $2 \times (12/40) = 24/40$ of the jelly beans. That leaves $1 – [(12/40) + (24/40)] = (40/40) – (36/40) = 4/40 = 1/10$ of the jelly beans, which Drew ate last. The ratio of the number of jelly beans Drew ate to the number Pat ate is $(4/40)/(4/40) = 4/4 = 1/6$.

198. Instead of thinking about each of Randolph’s favorite numbers as a whole, let us think of each digit separately. We are looking for three digits that have a sum of 9, with no two digits the same and the first digit can’t be zero. Let us represent the sum of 9 with diamonds. We need to separate these diamonds into three different sections, so we need two partitions, as shown.

- $\{\bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet\} = 324$
- $\{\bullet, \bullet, \bullet, \bullet, \bullet, \bullet\} = 126$
- $\{\bullet, \bullet, \bullet, \bullet, \bullet\} = 261$

No matter how we rearrange these diamonds and partitions, they will always correspond to one set of three numbers with a sum of 9. However, one diamond, indicated by the unfilled diamonds in the figure, must remain in the first partition and cannot be rearranged. It must remain there, so that the first number is at least 1. We can rearrange the remaining eight diamonds and two partitions. We must choose two places for the partitions out of 10 total places, and the rest will be diamonds. We can do this in $\begin{pmatrix} 10 \\ 2 \end{pmatrix}$ ways, which is $10 \times 9 / 2 = 45$ ways. However, we must remove the cases in which two of the numbers are the same. This happens when the digits are 117, 225 and 441, each of which can be rearranged in three ways, and when the digits are 333 and 900, each of which can be arranged in only one way. We need to subtract a total of $3 + 3 + 1 + 1 = 11$ ways from our initial 45 ways, so our answer is $45 – 11 = 34$ integers.

199. The slope-intercept form of a linear equation is $y = mx + b$, where $m$ is the slope and $b$ is the $y$-intercept. The slope of the line containing points $(2, 0)$ and $(–4, –3)$ is $m = (–3 – 0)/(4 – 2) = –3/2$. We can solve for $b$ by using this slope and either one of the points and substituting these values for $x$, $y$ and $m$ in the equation. Using the point $(2, 0)$, we get $0 = (–3/2) \times 2 + b$, which simplifies to $0 = –3 + b$, and is $b = 3$. Therefore, the $y$-intercept is $(0, 3)$.
200. The five integers that we have are often referred to as the minimum, lower quartile, median, upper quartile and maximum. In an ordered set of 11 integers, these values would be spaced out as follows: 12, ___, 29, ___, 35, ___, ___, ___, 39, ___, 52. We want to fill the blanks with six integers that will maximize the mean while not changing any of these five statistics. To maximize the mean, the 1st blank must be 29 since it's the largest integer that is greater than 12 but less than or equal to 29. Similarly, the 2nd and 3rd blanks must both be 35, since it's the largest integer that is greater than 29 but less than or equal to 35. Using the same reasoning, the 4th and 5th blanks should each be 39, and the final blank should be 52. This set of integers yields the greatest mean, which is \[(12 + 2(29) + (3 \times 35) + (3 \times 39) + (2 \times 52))/11 = 36.\]

Workout 7

201. This problem may require some exploration. The prime factorization of 720 (the volume) is \(2^4 \times 3^3 \times 5^1\). This number has 5 \(\times\) 3 \(\times\) 2 = 30 factors, so there are a lot of possible sets of three integer side lengths for the rectangular prism. Let's suppose the edge lengths are \(a\), \(b\) and \(c\). Then, using the surface area formula, we have \(666 = 2ab + 2ac + 2bc\). Dividing both sides of this equation by 2 yields the equation \(333 = ab + ac + bc\). Now we have to select the values for our edge lengths \(a\), \(b\) and \(c\) from the prime factors of 720 so that this equation is true.

The fact that 333 is odd is very helpful. This means that one of the products we can see that \(A\) must be 9 to get us to a four-digit sum. Since \(A = 9\) and we carry a 1, the hundreds digit of the sum must be 0. In 205.

When two digits are added, the most that can “carry” from one place value to the next is 1. This means that \(R\) must be 1. Also, 204.

202. The sum of all 14 of Marco’s quiz scores must equal 14 times his quiz average, which is \(14 \times 81 = 1134\). Similarly, the sum of his first 10 quiz scores is \(10 \times 75 = 750\). The sum of the last 4 quiz scores must be 1134 – 750 = 384. If the sum of his scores on the last 4 quizzes is 384, his average on the last 4 must be 384 ÷ 4 = 96.

203. Let’s suppose that \(AB = 1\) unit. Then, by properties of 30-60-90 right triangles, the altitude of \(\Delta ABC\) is \(\sqrt{3}/2\) units, which makes its area equal to \((1/2) \times 1 \times (\sqrt{3}/2) = \sqrt{3}/4\) units². By properties of 45-45-90 right triangles, the altitude of \(\Delta ABD\) is \(1/2\) unit and its area is \((1/2) \times 1 \times (1/2) = 1/4\) units². The area of quadrilateral \(ACBD\) is the difference between these two areas, which is \((\sqrt{3}/4) – (1/4) = (\sqrt{3} – 1)/4\) units². The desired ratio is \([(\sqrt{3} – 1)/4]/(1/4) = [(\sqrt{3} – 1)/4] × 4 = (\sqrt{3} – 1) = 0.73\), to the nearest hundredth.

204. The center of a circle is at the midpoint of any diameter. If we take the average of the \(x\)-coordinates and the average of the \(y\)-coordinates, we get the midpoint of the segment. So, \(x = (–3 + 11)/2 = 8/2 = 4\) and \(y = (–2 + –10)/2 = –12/2 = –6\). Thus, the center is at \((4,–6)\).

205. When two digits are added, the most that can “carry” from one place value to the next is 1. This means that \(R\) must be 1. Also, we can see that \(A\) must be 9 to get us to a four-digit sum. Since \(A = 9\) and we carry a 1, the hundreds digit of the sum must be 0. In the units place, we find that we cannot allow a carry, since that would make \(T = Y\). Therefore, we need an \(E\) and a \(T\) with a sum of 2, 3, 5 or 7. We soon see that sums of 2 and 3 don’t work. Then no combination to make 5 works. Eventually we see that \(E = 4, T = 3\) and \(S = 7\) works. Now we have the sum as shown. The desired value is \(A + T + E + O = 9 + 3 + 4 + 0 = 16\).

206. If the 2 students who completed a marathon are 2.5% of the population, then there are a total of \(2 ÷ 0.025 = 80\) ninth-graders.

207. The location of point \(P\) on the line that passes through points \(A(5, 0)\) and \(B(0, 2)\) must be \(1/3\) of the distance from point \(A\) to point \(B\), so as to partition segment \(AB\) in a 1:2 ratio. The \(x\)-coordinate of \(P\) must be \((1/3) \times 5 = 5/3\) and its \(y\)-coordinate must be \((1/3) \times 2 = 2/3\), so the coordinates of point \(P\) are \((10/3, 2/3)\). Now we can substitute these values into the equation \(y = kx\) to solve for \(k\). We get \(k = y/x = (2/3)/(10/3) = 2/10 = 1/5\).

208. The perimeter of the whole shaded portion is half the circumference of the circle with radius of length \(r\) units plus the circumference of two halves of the circles with diameter of length \(r\) units. This amounts to \((1/2) \times (\pi \times 2r) + (2 \times (1/2) \times (\pi \times r)) = \pi r + \pi r = 2\pi r\) units.

209. Let’s work this problem backward. After Arlene finished shopping at the three stores, she had $2 left over. Before she spent the $6 at the third store, she must have had $8. This $8 must have been the other 40% that was left after she spent 60% of her money, so she must have had $8 ÷ 0.4 = $20 when she left the first store. Before she spent the $5 at the second store, she must have had $25. This $25 was the other 50% that was left after she spent 50% of her money, so she must have had $25 ÷ 0.5 = $50 when she left the first store. Before she spent the $4 at the first store, she must have had $54. This $54 was the 60% that was left after she spent the first 40% of her money, so she must have started with $54 ÷ 0.6 = $90 dollars.

210. The first day on which Judith, Mark and Kelly all received a gift was the 49th day of the year. Since the least common multiple of 6, 5 and 4 is 60, they all received a gift every 60 days after that. We can add 60 to 49 five times to arrive at the 349th day of the year. So, including the first time, there were a total of 6 days in the year when all three received a gift.

Warm-Up 15

211. According to the survey, \(1/4 + 1/5 + 1/3 = (15 + 12 + 20)/60 = 47/60\) of the students have at least one pet, grandparent or baby living in their homes. The other \(1 – 47/60 = 60/60 – 47/60 = 13/60\) of the students have no pets, grandparents or babies living in their homes. Therefore, the least possible number of students who have none of these living in their homes is \(13/60 \times 1500 = 325\) students.

212. Now \(5^2 + 2 = 25 + 2 = 27\), which has a units digit of 7. Then \(7^2 + 2 = 49 + 2 = 51\), which has a units digit of 1. Next we have \(1^2 + 2 = 1 + 2 = 3\), which has a units digit of 3. Finally, \(3^2 + 2 = 9 + 2 = 11\), which has a units digit of 1. So we can see that after the first two terms, the sequence alternates between 1 and 3, with 1 for the odd-numbered terms and 3 for the even-numbered terms. Since 100 is even, we can conclude that the 100th term in the sequence is 3.
213. If we place all the given numbers in the appropriate regions of a Venn diagram such as the one shown here, we see that the only unknown quantity is the number of students who participate in baseball only. Therefore, we can subtract the sum of the other regions from 78 to obtain this quantity. Together the other regions have a sum of $3 \times 7 + 4 \times 2 + 5 + 1 = 22$. That means the other $78 - 22 = 56$ students must play baseball only.

214. If we cross out all locations that are on the sight lines for each of the soldiers already placed, we are left with locations A5, B4 and E5. Location A5 shares a sight line with both B4 and E5, so we can’t use that location. Thus, the two other soldiers can only be at locations B4 and E5.

215. Using the Pythagorean Theorem, we can find the length of the hypotenuse, $h$, of our right triangle as follows: $h^2 = (3/2)^2 + (20/3)^2 \rightarrow h^2 = 9/4 + 400/9 \rightarrow h^2 = 81/36 + 1600/36 \rightarrow h^2 = 1681/36 \rightarrow h^2 = 41/6$. For the perimeter of the triangle, we have $x = 3/2 \times 20/3 + 41/6 = 9/6 + 40/6 + 41/6 = 90/6 = 15$ units. For the area of the triangle, we have $y = (1/2) \times (3/2) \times (20/3) = 5$ units. Thus, the value of $x^2 - y^2 = 15^2 - 5^2 = 225 - 25 = 200$.

216. Writing the complete expansion of $(x + y + z)^3$ would be quite tedious, but we don’t need the complete expansion. We only need the coefficient of the $x^2y^2z^2$ term. Let’s conceptualize this a bit and try to approach the problem strategically. If we write out $(x + y + z)^3$ as $(x + y + z)(x + y + z)(x + y + z)$, we can see a way to restate the problem. We want to know how many ways we can choose two $x$s, two $y$s and two $z$s from the six sets of parentheses, because all of these will be like terms that combine to make the total number of $x^2y^2z^2$ terms. There are “6 choose 2” ways to pick two $x$s then “4 choose 2” ways to pick two $y$s, and then the two $z$s are forced to be from the remaining sets of parentheses. Thus, the coefficient of the $x^2y^2z^2$ term in the expansion is $\binom{6}{2} \cdot \binom{4}{2} \cdot \binom{2}{2} = 15 \cdot 6 \cdot 1 = 90$.

217. There are 90 two-digit integers from 10 to 99. If we consider zero to be a multiple of any whole number, then we can count all 10 integers in the teens, all even integers in the twenties, all multiples of 3 in the thirties, etc. There are $10 + 4 + 3 + 2 + 2 + 2 + 2 + 2 + 2 = 32$ integers with a units digit that is a multiple of the tens digit. The probability that such an integer is randomly selected is $32/90 = 16/45$.

218. The country of Quaternion seems to use a base four system for its coins. We can use the last letter of each coin name to specify their values as follows: $a = 10, b = 4a, c = 16a, d = 64a, e = 256a$ and $f = 1024a$. To come up with a value of 2012 in the fewest coins, we will write 2012 as a sum of these powers of 4 and then convert to the letters and the coins. We see that $2012 = 1024 + (3 \times 256) + (3 \times 64) + 16 + (3 \times 4) = f + 3e + 3d + c + 3b$. A citizen of Quaternion could make 2012 quas with 11 coins—1 quaf, 3 quaes, 3 quads, 1 quac and 3 quabs.

219. The values of $m$ and $n$ must both be even or both be odd. If we consider all pairs of $(m, n)$ with $m < n$ and both even, we get a triangular arrangement, as shown. There will be 49 ordered pairs along the bottom row, so there are $49 \times 50 = 2450$ pairs. There will also be 1225 odd pairs, so there are $2 \times 1225 = 2450$ pairs, in all.

220. Triangles AFG, ADE and ACB each have a 90° angle and share angle A, so the third angle of each triangle must be congruent. Therefore, AAFG, DADE and DABC are all similar. This means that the ratios of corresponding sides are equal. In particular, $AG:GF = AE:ED = AB:BC$, so $4/A = AE/2x = AB/3x$, and therefore, we know that $AE = 8$ units and $AB = 12$ units. We need to find the value of $x$. A key step is to notice that quadrilateral DEFG is a kite, so $DG = DE = 2x$ units. Thus, $AD = 4 + 2x$ units. We also have that $AD = \sqrt{8^2 + (2x)^2} = \sqrt{64 + 4x^2}$ units. To solve the equation $4 + 2x = \sqrt{64 + 4x^2}$, we square both sides, which gives us $16 + 16x + 4x^2 = 64 + 4x^2$. Subtracting $4x^2$ from both sides, we get $16 + 16x = 64$, which means that $16x = 48$ and $x = 3$. Now we can determine that $\Delta AFG$ is a 3-4-5 right triangle and $BC = 9$. Since we know $AB = 12$, we can determine that $\Delta ABC$ is a 9-12-15 right triangle. Because $CA = 15$, $FE = 3$ and $AF = 5$, we know $CE = 15 - 3 = 12$ units.

Warm-Up 16

221. If the two-digit number is $AB$, then its value is $10A + B$. When the digits are reversed, the number would be $BA$ with a value of $10B + A$. We are told that $BA$ is 75% greater than $AB$, so we have the equation $10B + A = 1.75(10A + B)$, which can be rewritten as $10B + A = (7/4)(10A + B)$ and simplified to $40B + 4A = 70A + 7B \rightarrow 33B = 66A \rightarrow B = 2A$. This means that any number with a units digit equal to twice the tens digit will work. Thus, 12, 24, 36 and 48 are the 4 possible two-digit integers.

222. The altitude of the smaller pyramid is not only half as long as the altitude of the larger pyramid, but the dimensions of its base also are half the dimensions of the larger pyramid’s base. The smaller pyramid’s volume must be $(1/2)^2 \times (1/2) = 1/8$ the volume of the larger pyramid, and $(1/8) \times 64 = 8$ cm$^3$.

223. There are several ways to approach this problem algebraically, one of which follows. If we find the common denominator for each side, we can set up a proportion. We have $x + 4/x = (x^2 + 4)/x$, and $y + 4/y = (y^2 + 4)/y$, so $(x^2 + 4)/x = (y^2 + 4)/y$. We can cross multiply to get the equation $y(x^2 + 4) = x(y^2 + 4)$. When we distribute on each side of the equation, the result is $x^2y + 4y = xy^2 + 4x$. Now if we subtract $xy^2$ from each side and subtract $4y$ from each side, we have $x^2y - xy^2 = 4x - 4y$. Factoring out $xy$ on the left and factoring out 4 on the right, the result is $xy(x - y) = 4(x - y)$. Therefore, $xy = 4$.

224. There are 2 shaded regions that are $3/4$ of $1/4$ of the largest square. Thus, the total shaded area is $2 \times (3/4) \times (1/4) = 3/8$ of the total area of the largest square. Therefore, the probability that the dart will hit the shaded region is $3/8 = 37.5\%$.

225. There are “8 choose 2” ways to choose the 2 girls, which yields $8 \times 7 / 2 = 28$ pairs of girls. Similarly, there are “11 choose 2” ways to choose the 2 boys, which is $11 \times 10 / 2 = 55$ pairs of boys. Each of the 28 pairs of girls can be matched with each of the 55 pairs of boys, so there are $28 \times 55 = 1540$ possible dance teams.

MATHCOUNTS 2011-2012 83
235. In the game of Krypto, there are $3 \times 10 = 30$ cards from 1 to 10, $2 \times 7 = 14$ cards from 11 to 17 and $1 \times 8 = 8$ cards from 18 to 25. That's $10^2 = 100$ and $11^2 = 121$, the value of $\sqrt{113}$ must be between 10 and 11. The desired sum is $10 + 11 = 21$, so we must have $x = 80$, which is the old average. Therefore, the new average is $80 + 5 = 85$ points.

236. Each of the smaller blocks has a volume of $3 \times 1 \times 1 = 3$ in$^3$. The volume of the interior of the larger box is $5 \times 5 \times 10 = 250$ in$^3$. One would figure the larger box will hold $250 + 3 = 83$ of the smaller blocks. But upon closer inspection and attempting to arrange the 3-in$^3$ blocks inside the box, we see that the dimensions of the box will not accommodate 83 blocks. As shown, 50 of the blocks can fit horizontally in 10 stacks, each 5 blocks high. Another 20 blocks will fit vertically in 2 columns of 10. Finally, 12 blocks can fit horizontally, but rotated 90 degrees in 6 stacks, each 2 blocks high. There is a $2^2 \times 2^2 \times 1^2$ space that remains unfilled. That's a total of $50 + 20 + 12 = 82$ blocks.

237. The three index cards in the center of the structure form an equilateral triangle, so the angles must each measure 60°. Since the structure rests on a flat, level surface, the angles on either side of the 60° angle in the middle must add up to $180 - 60 = 120°$. If we let the measure of one of the angles be $a$ and the measure of the other angle be $b$, then we know that $a + b = 120°$. Also, since all the index cards are the same length, the triangles on the left and right must each be isosceles. This means we have another angle of $a$ degrees in the left triangle and another angle of $b$ degrees in the right triangle. The sum of the angles of any triangle is always 180°, so the sum $2a + x + 2b + y = 2 \times 180$. This can be rewritten as $2(a + b) + x + y = 360$. Substituting 120 for $a + b$, we get $240 + x + y = 360$, so $x + y = 120°$.

238. If we "drop a perpendicular" from the vertex angle of the triangle, it will intersect the base at its midpoint, dividing the isosceles triangle into two congruent right triangles. Each of these two triangles is a 5-12-13 right triangle. The area of the entire isosceles triangle is $\frac{1}{2} \times 10 \times 12 = 60 \text{ cm}^2$. Let's say that the side length of the square is $x$ units. Then the small right triangle on the left is similar to the 5-12-13 triangle and has leg lengths of $x$ and $5 - x/2$. We can set up the following proportion: $(5 - x/2)/x = 5/12$. Cross multiplying yields the equation $60 - 6x = 5x$. So $60 = 11x$, and $x = 60/11$. Thus, the area of the square is $(60/11)^2 = 3600/121 \text{ cm}^2$. Subtracting this area from the area of the entire triangle, we get $60 - 3600/121 = 7260/121 - 3600/121 = 3660/121 = 30.25 \text{ cm}^2$.

239. If we multiply both sides of the equation by $x + 1$, we get $x^2 + ax + 6 = x^2 + (b + 1)x + b$. For these two expressions to be equal, we must have $b = 6$ and $a = b + 1 = 6 + 1 = 7$. The value of $6a - 7b$ is $6 \times 7 - 7 \times 6 = 42 - 42 = 0$.

**Workout 8**

231. Let's say Matt's average for the five tests is $x$. Then the new average is $(5x - 60) / 4$, which we are told is 5 more than the old average, or $x + 5$. Now we have the equation $(5x - 60) / 4 = x + 5$. Multiplying both sides of the equation by 4, we get $5x - 60 = 4x + 20$. Simplifying, we get $x = 80$, which is the old average. Therefore, the new average is $80 + 5 = 85$ points.

232. The horizontal distance between the two points is $4 - (-4) = 8$ units. The vertical distance is $2 - (-5) = 7$ units. These are the legs of a right triangle, so the distance, $d$, between the points can be found using the Pythagorean Theorem as follows: $d = \sqrt{8^2 + 7^2} = \sqrt{64 + 49} = \sqrt{113}$. Since $10^2 = 100$ and $11^2 = 121$, the value of $\sqrt{113}$ must be between 10 and 11. The desired sum is $10 + 11 = 21$.

233. The decimal value of $\frac{8}{81}$ is 0.098765337. If we were to write out 2007 digits, we would have 223 complete sets of the nine digits in the pattern. The 2012th digit must be the fifth digit in the pattern, which is 6.

234. Tito wants to pay 20% of the bill as a tip and he has to pay 6% of the bill in taxes, so he will pay an extra 26% of the bill. We can get the total amount he will pay by multiplying the bill by 1.26, which is $1.26 \times $19.50 = $24.57$.

235. In the game of Krypto, there are $3 \times 10 = 30$ cards from 1 to 10, $2 \times 7 = 14$ cards from 11 to 17 and $1 \times 8 = 8$ cards from 18 to 25. That's $30 + 14 + 8 = 52$ cards in all. Since there are 30 cards containing a number from 1 to 10, the probability that the first card selected at random is 10 or less is $\frac{30}{52}$. Then the probability that the second randomly selected card will be 10 or less is $\frac{29}{51}$, and so forth. So the probability that five cards selected randomly are 10 or less is $(\frac{30}{52}) \times (\frac{29}{51}) \times (\frac{28}{50}) \times (\frac{27}{49}) \times (\frac{26}{48}) = 0.055$.

236. Subtracting $2x$ from each side yields $3x - 30 = 54$. Then we can add 30 to each side to get $3x = 84$. Once we divide each side by 3, we are left with $x = 28$. If we square each side of the equation, we see that $x = 784$.

237. Since the ratio of the dimensions of the rectangular prism is 1:2:3 and the shortest edge is 2 ft, the dimensions must be 2 ft, 4 ft and 6 ft. The longest distance between any two vertices in the prism is the space diagonal, which can be the hypotenuse of a right triangle whose short leg has length 2 ft. The long leg is the diagonal of a 2-by-6 rectangle and has a length of $\sqrt{6^2 + 2^2} = \sqrt{40}$. That means the space diagonal has length $\sqrt{(\sqrt{40})^2 + 4^2} = \sqrt{40 + 16} = \sqrt{56} = 7.48$ ft.

238. If we "drop a perpendicular" from the vertex angle of the triangle, it will intersect the base at its midpoint, dividing the isosceles triangle into two congruent right triangles. Each of these two triangles is a 5-12-13 right triangle. The area of the entire isosceles triangle is $\frac{1}{2} \times 10 \times 12 = 60 \text{ cm}^2$. Let's say that the side length of the square is $x$ units. Then the small right triangle on the left is similar to the 5-12-13 triangle and has leg lengths of $x$ and $5 - x/2$. We can set up the following proportion: $(5 - x/2)/x = 5/12$. Cross multiplying yields the equation $60 - 6x = 5x$. So $60 = 11x$, and $x = 60/11$. Thus, the area of the square is $(60/11)^2 = 3600/121 \text{ cm}^2$. Subtracting this area from the area of the entire triangle, we get $60 - 3600/121 = 7260/121 - 3600/121 = 3660/121 = 30.25 \text{ cm}^2$.

239. If we multiply both sides of the equation by $x + 1$, we get $x^2 + ax + 6 = x^2 + (b + 1)x + b$. For these two expressions to be equal, we must have $b = 6$ and $a = b + 1 = 6 + 1 = 7$. The value of $6a - 7b$ is $6 \times 7 - 7 \times 6 = 42 - 42 = 0$.
240. Initially the ladder forms a 45-45-90 right triangle. The ladder is the hypotenuse with length 6 ft, and the floor and wall are the two congruent legs of the right triangle. Using properties of 45-45-90 triangles, we see that the top of the ladder rests against the wall at a distance of \(6/\sqrt{2} = 3\sqrt{2}\) ft above the floor. This is also the distance from the base of the ladder to the wall. When this distance is decreased by 2/3, the new distance from the base of the ladder to the wall is \(3\sqrt{2}/3 = \sqrt{2}\). Using the Pythagorean Theorem, we see that the top of the ladder now rests \(v(6^2 - (\sqrt{2})^2) = v(36 - 2) = v(34)\) ft from the floor. Thus, the top of the ladder has moved farther up the wall a distance of \(v(34) - 3\sqrt{2} = 1.59\) ft.

**Warm-Up 17**

241. There are four ways to get a sum of 15 using four different whole numbers from 1 to 7. They are \([7 + 5 + 2 + 1],[7 + 4 + 3 + 1],[6 + 5 + 3 + 1]\) and \([6 + 4 + 3 + 2]\). The first three sums can be arranged together, as can the last three sums. If we use the first three, then \(A\) is replaced with 1. If we use the last three, then \(A\) is replaced with 3. The sum of all possible replacements for \(A\) is \(1 + 3 = 4\).

242. Combined, the area of the shaded regions in the left upper and lower 4-by-4 squares is equal to the area of one 4-by-4 square, which is \(4 \times 4 = 16\) units\(^2\). The shaded region of the lower right 4-by-4 square is \((3/4) \times 16 = 12\) units\(^2\). Finally, the circle in the upper right 4-by-4 square has a radius of \((1/2) \times 4 = 2\) units and an area of \(\pi \times 2^2 = 4\pi\). Thus, the area of the shaded region surrounding the circle is 16 – 4\(\pi\) units\(^2\). Therefore, the total area of the shaded regions is \(16 + 12 + (16 - 4\pi) = 44 - 4\pi\) units\(^2\).

243. The mean of an arithmetic sequence is the middle number when there are an odd number of terms, so we know the middle number is 18. Let’s call the common difference between terms \(d\). Then the five numbers are \(18 - 2d, 18 - d, 18 + d, 18 + 2d\). The sum of the squares of these five numbers is \((32^2 - 72 + 4d^2) + (32^2 - 36d + d^2) + (32^2 + 36d + 4d^2) + (32^2 + 72d + 4d^2) + (32^2 + 100d^2) = 5 \times 32^2 + 10d^2\). We divide this by 5 to get a mean of \(32^2 + 2d^2\), which is known to be 374. This means that \(2d^2 = 374 - 32^2 = 72\rightarrow 2d = 12\). The largest of the five original numbers is \(18 + 2 \times 5 = 28\).

244. We need to determine how many sets of prime numbers, not necessarily distinct, have a sum of 13. We need to consider sets containing prime numbers between 2 and 13. The 9 numbers for which the sum of the prime factors is 13 are shown.

245. Let’s work this problem backward. At the end, Xavier, Yvonne and Zeena all have 48 marbles, so let \(x = 48, y = 48\) and \(z = 48\). Before Zeena doubled Xavier’s and Yvonne’s marbles, it was \(x = 24, y = 44\) and \(z = 96\). Before Yvonne doubled Xavier’s and Zeena’s marbles, it was \(x = 12, y = 84\) and \(z = 48\). Before Xavier doubled Yvonne’s and Zeena’s marbles, the starting quantities were \(x = 78, y = 42\) and \(z = 24\). Xavier had \(78 - 48 = 30\) fewer marbles at the end than he started with.

246. The $\text{S14}$ represents the extra \(7 - 5 = 2\) parts of the phone bill that Barbara had beyond Tina’s phone bill. If $\text{S14}$ was 2 parts, then Barbara’s bill must have been \(7 \times 5 = \$49\).

247. The least possible sum of four different positive integers is \(1 + 2 + 3 + 4 = 10\). To get to a sum of 13, we have to add 3 more to these numbers in such a way that they remain different. There are only 3 ways this can be done. They are \([1 + 2 + 3 + 7],[1 + 2 + 4 + 6]\) and \([1 + 3 + 4 + 5]\). For each of these possible sums, there are 4! = \(4 \times 3 \times 2 \times 1 = 24\) ways the numbers can be placed in the 4 boxes. Thus, there are a total of \(3 \times 24\) = 72 ways in which four different positive integers with a sum of 13 can be arranged in the boxes.

248. Solving this problem requires some tricky algebra. Let the three numbers be \(a, b,\) and \(c\). From the information given, we have \(a + b + c = 5, a^2 + b^2 + c^2 = 29\) and \(abc = -10\). If we square each side of the first equation, we get \((a + b + c)^2 = 25\rightarrow a^2 + b^2 + c^2 + 2ab + 2bc + 2ac = 25\). Now let’s substitute 29 for \(a^2 + b^2 + c^2\) in this equation to get \(29 + 2ab + 2bc + 2ac = 25\). Subtracting 29 from each side and dividing the equation by 2 yields \(ab + bc + ac = -2\). Let’s now multiply each side of this equation by \(b\) to get \(abc + b^2 + ac^2 = -2c\). We can replace \(abc\) with \(-10\) to get \(-10b + b^2 + ac^2 = -2c\). Now let’s factor out \(c\) on the left side of the equation to get \(-10c + c(b + a) = -2c\). If \(a + b + c = 5\) then \(a + b = 5 - c\). Thus, we can replace \(b + a\) with \(5 - c\) to get \(-10c - 5c - c = -2c\). Distributing \(c\) and combining like terms yields \(-10c - 5c - c = -2c\). If we set this cubic equal to 0, the result is \(c^2 - 5c^2 - 2c\). Factoring this equation yields \(c^2(c - 5) - 2(c - 5) = 0\rightarrow (c^2 - 2)(c - 5) = 0\). Now we see that the three solutions are \(-\sqrt{2},\sqrt{2}\) and 5, the least of which is \(-\sqrt{2}\).

249. Let’s say the total value of the quarters and dimes is \(T\). Then \(T = 25q + 10d\), where \(q\) is the number of quarters and \(d\) is the number of dimes. When 10% more quarters are added, the value of \(T\) increases by 7.5%, or \(1.075T = T + 25q + 10d\). Let’s rewrite this with fractions. Ten percent is \(1/10\) and \(7.5\%\) is \(3/4\), so we have the following equation: \((43/40)T = (11/10)(25q + 10d)\). Subtracting the original value of \(T\), we have \((43/40)(25q + 10d) = (11/10)(25q) + 10d\). Multiplying out both sides and simplifying, we get \(2(25q)/40q + (43/40)d = (55/2q + 10d - (43/40)/d)\). Subtracting \((220/8)q - (215/8)q) → (3/4)d = (58/8)q → a/d = (3/4)(58/8) = q/d = (3/4) × (8/5) = 6/5\).

250. After transforming the trapezoid, the resulting shape is a frustum, which is simply a cone with the tip removed. We can determine the volume of the frustum by determining the volume of the cone less the volume of the portion that is removed to form the frustum. If we drop a perpendicular from point \(B\) down to \(CD\), as shown, and label the point of intersection \(E\), then we have right \(\triangle BCE\) with hypotenuse of length 15 units and one leg of length 16 – 4 = 12 units. This is a multiple of the 3-4-5 Pythagorean triple 9-12-15, so the length of segment \(BE\) must be 9 units. If we extend \(AD\) and \(BC\) until they intersect at point \(F\), we create \(\triangle FCD\) and \(\triangle FA\) with sides of length 12 units, 16 units and 20 units, and 3 units, 4 units and 5 units, respectively. (This is because \(AB = 4\) and \(\triangle ABC = \triangle FBA\).) Now let’s revolve \(\triangle FCD\) around \(FC\). Recall that the volume of a cone is \(1/3\pi r^2h\) the area of its base. Therefore, the volume of this cone is \(1/3 \times 12 \times \pi \times 15^2 = 1024\pi\) units\(^3\). Next we revolve \(\triangle FBA\) around \(FA\). The volume of this cone is \(1/3 \times 3 \times \pi \times 4^2 = 16\pi\) units\(^3\). Thus, the volume of the frustum is \(1024\pi - 16\pi = 1008\pi\) units\(^3\).
251. It is safe to assume that the greatest possible product will be produced by a two-digit number times a three-digit number, so let’s call those numbers AB and XYZ. The value of AB is 10A + B and the value of XYZ is 100X + 10Y + Z. The product (10A + B)(100X + 10Y + Z) = 1000AX + 100AY + 10BX + 10A2 + 108Y + BZ. First let’s compare the way the digits A and X are used. They each get multiplied by 1000 and by each other, which is symmetric. They also each get multiplied by 100 and by another digit, but A is also multiplied by 10Z; therefore we want A to be the largest number of the set, and X should be the next largest. Thus, we determine that A = 8 and X = 7. We can now write (1000 × 8 × 7) + (100 × 8 × Y) + (100 × 7 × B) + (10 × 8 × Z) + 108Y + BZ = 56,000 + 800Y + 700B + 80Z + 108Y + BZ. Now, we can see that to get the largest product, Y should be the next greatest value, followed by B and then Z. So we have Y = 5, B = 3 and Z = 1. The two numbers are 83 and 751, and their product is 62,333.

252. Since the total area is 400 ft² and the ratio of the areas of the two smaller rectangles is 3 to 1, they must have areas of 300 ft² and 100 ft². The least possible perimeter for the smaller rectangle would be 4 × 10 = 40 ft, from a 10 × 10 square.

253. A set of seven positive integers with a mean of 13 must have a sum of 7 × 13 = 91. Since the integers must be positive and different, the smallest possible value of the median is 4. The numbers could be 1, 2, 3, 4, and any combination of three different positive integers adding up to 91 – (1 + 2 + 3 + 4) = 81. To find the largest possible median, we should first allow for 1 + 2 + 3 = 6 at the low end. That leaves 91 – 6 = 85 to be distributed to the other four integers. We will need another 1 + 2 + 3 = 6 to make the 5th, 6th and 7th integers greater than the 4th integer, so that leaves 85 – 6 = 79 to divide by 4. Algebraically, we can solve n + (n + 1) + (n + 2) + (n + 3) < 85 for the greatest integer value of n that satisfies the inequality. This simplifies to 4n < 79, and 19 is the greatest integer value of n. The desired difference is 19 – 4 = 15.

254. The value of 123 (base b) is 1 × b² + 2 × b + 3 × 1 and this must equal 363. Algebraically, this can be written as b² + 2b + 3 = 363. We set the equation equal to zero to get b² + 2b – 360 = 0. To factor this quadratic, we look for two factors of 360 that differ by 2. They are 18 and 20, so the equation can be written as (b + 20)(b – 18) = 0. The solutions to this are b = –20 and b = 18. Only the positive solution makes sense here, so the base must be 18.

255. Let GC = y, so DG = 2y and DC = 3y. Now let the length of the altitude of ΔDHG be h. Then the altitude of ΔGCF has length 2h and the altitude of rhombus ABCD has length 3h. The area of rhombus ABCD is then 3y × 3h = 9y² and is known to be 180 units². This means that yh = 180 ÷ 9 = 20. The area of ΔDHG is 1/2 × 2y × h = yh = 20 units², and the area of ΔGCF is (1/2) × y × 2h = yh = 20 units². The areas of ΔAEH and ΔEBF are also each 20 units². This means that rhombus EFGH must have an area of 180 – (4 × 20) = 100 units².

256. Let’s substitute the solution x = u into the first equation and x = 1/u into the second equation. The resulting equations are ru² + su + t = 0 and (2 + a)u² + 5/u + (2 – a) = 0. We rewrite the second equation with a common denominator and get (2 + a)u² + 5u/u² + ([2 – a]u²)/u² = 0. Multiplying both sides of the equation by u², we get (2 + a)u² + 5u + (2 – a)u² = 0. Since our first equation and this last equation are both equal to zero, we can set them equal to each other to get ru² + su + t = (2 + a)u² + 5u + (2 – a)u². Both of these expressions are in terms of u, so the coefficients of the same degree must be equal. This means that r = 2 – a, s = 5 and t = 2 + a. Thus, the sum r + s + t is 2 – a + 5 + 2 + a = 9.

257. There are C₂₅ = 22 × 21 × 20 × 19 / 4 × 3 × 2 × 1 ways to choose a committee of 4 from a group of 22. Similarly, there are C₂₀ = 22 × 21 × 20 × 19 / 5 × 4 × 3 × 2 × 1 ways to choose a committee of 5 from a group of 22. The ratio of the number of ways to choose a 4-person committee to the number of ways to choose a 5-person committee is C₂₀ / C₂₅ = (22 × 21 × 20 × 19 / (5 × 4 × 3 × 2 × 1)) / (22 × 21 × 20 × 19 / 4 × 3 × 2 × 1) = 5. So there are 5 ways to choose a 4-person committee to the number of ways to choose a 5-person committee.

258. Quadrilateral WXYZ is a translation of ABCD 11 units to the right and 7 units down, as shown. The coordinates of point Z are (5, -7) and the sum of the coordinates is 5 + (-5) = 0.

259. Substituting a and c = 49 – a into the Pythagorean Theorem and simplifying yields (49 – a)² – a² = b² → 2401 – 98a + a² = b² → 2401 – 98a = b² → 49(49 – 2a) = b². This means that 49 – 2a is an odd number, in which case the candidates are 1, 9, 25 and 49. If 49 – 2a = 9, solving for a yields -2a = -40 → a = 20. So we have a = 20, b = 21 and c = 29. The area of this triangle is (1/2) × 20 × 21 = 210 units². If 49 – 2a = 25, solving for a yields -2a = -24 → a = 12. So we have a = 12, b = 35 and c = 37. The area of this triangle is (1/2) × 12 × 35 = 210 units². Finally, 49 – 2a = 49 doesn’t work since it results in a = 0. The area of each of the two triangles that satisfy our conditions is 210 units².

260. This problem may prove easier to solve if we can figure out the shape of the coordinate plane. Let’s suppose the left, rear corner of the base, which is point P, is situated at the origin with coordinates (0,0,0). Since all the edges have length 2 units, the other vertices would be located as follows: F(2, 0, 0), G(2, 0, 2), H(0, 0, 2), A(0, 2, 0), B(2, 0, 0), C(2, 2, 2), D(0, 2, 2), M(0,0,1) and N(1,0,0). If we let point K be the midpoint of MN, the original tetrahedron is divided into two congruent tetrahedrons with common base ΔACK. Since MN is the hypotenuse of an isosceles right triangle with legs of length 1 unit, its length is V2 units. Thus, each of these two smaller tetrahedrons has height h = MN = NK = V2/2 units. Now to determine the area of ΔACK, we first consider rectangle AGCE. Since AC is the diagonal of one of the faces of the cube, it is also the hypotenuse of a 45-45-90 right triangle with legs of length h units. That means AC = 2V2 units and rectangle AGCE has an area of 4V2 units². Next we determine the area of ΔAEK to be (1/2) × (V2/2) × 2 × V2 units², and the area of ΔGCA is (1/2) × (3V2/2) × 2 = 3V2 units². If we subtract these two areas from the area of rectangle AGCE, the result is the area of ΔACK. The area of ΔACK is 4V2 – (V2/2) – (3V2/2) = (V2/2) – (3V2/2) = (4V2/2) = 2V2 units². The volume of each of the smaller tetrahedrons is (1/3) × (2V2) × (V2/2) = 2/3 units³. Therefore the volume of the large tetrahedron is 2/3 + 2/3 = 4/3 units³.
**Workout 9**

261. When a 6-8-10 triangle is revolved about the side of length 10 units, the resulting solid can be thought of as two cones that share a base, as shown. The radius, \( r \), of the circular base is the altitude of \( \triangle ABC \) drawn perpendicular to \( AC \). Since 6-8-10 is a Pythagorean triple, we know we have a right angle between the legs of lengths 6 units and 8 units. Therefore, the area of the triangle is \( \frac{1}{2} \times 6 \times 8 = 24 \) units\(^2\). That means that \( \frac{1}{2} \times r \times 10 = 24 \) units\(^2\) and \( 5r = 24 \). So \( r = 4.8 \) units. This will serve as the radius of the base of each cone. Now we know the volume of the top cone is \( \frac{1}{3} \pi \times (4.8)^2 \times h_t \) and the volume of the bottom cone is \( \frac{1}{3} \pi \times (4.8)^2 \times h_b \). Together, these are \( \frac{1}{3} \pi \times (4.8)^2 \times (h_t + h_b) \). Thus, we know that \( h_t + h_b = 10 \) units, so the volume of the entire solid is \( \frac{1}{3} \pi \times (4.8)^2 \times 10 = 241.3 \) units\(^3\). Note: Using similar triangles, one could determine the lengths of \( h_t \) and \( h_b \) and find the volume of each of the two cones.

262. Let’s say the number of students wearing a blue jersey is \( B \) and the number wearing a red jersey is \( R \). Then either we can have two \( B \)s and one \( R \), or we can have one \( B \) and two \( R \)s. That means the number of ways to pick a team of 3 is \( C_2 \cdot C_1 + C_1 + C_1 = 0 \cdot 1 + 1 + 1 = 2 \) ways. Simplifying we have \( BR(B - 1) + BR(R - 1) = 50 \rightarrow BR(B - 1 + R - 1) = 50 \rightarrow BR(B + R - 2) = 50 \). Since both \( B \) and \( R \) are positive integers, we should try some factors of 50. It turns out that 2 and 5 work. So there are 2 students with blue jerseys and 5 with red, or vice versa. Either way, there must be 2 + 5 = 7 students in the gym class.

263. Each time the ticket price goes up by 4%, we can find the new value in one step if we multiply the old value by 1.04. Let’s say the price of game tickets was \( P \) dollars 13 years ago. Then the price now would be \( P \times 1.04^{13} \). The ratio of \( P \) to \( P \times 1.04^{13} \) is \( 1/1.04^{13} = 0.60 = 60\% \).

264. In a geometric progression, there is a constant ratio between consecutive terms; we will call it \( r \). Let’s say the length of the shorter leg of the right triangle is \( x \). Then the longer leg has length \( rx \), and the hypotenuse has length \( r^2x \). These three lengths must satisfy the Pythagorean Theorem, so we have \( x^2 + (rx)^2 = (r^2x)^2 \). If we factor out \( x^2 \), we get \( x^2(1 + r^2) = r^2x^2 \). Since we know that \( x \) is a positive length, it is safe to divide both sides of the equation by \( x^2 \). That leaves \( 1 + r^2 = r^2 \). Although this is a fourth degree equation, we can solve it like a quadratic if we substitute \( s = r^2 \). This gives us \( 1 + s = s^2 \). Since this does not factor, we can use the quadratic formula to find that the solutions are \( s = (1 \pm \sqrt{5})/2 \). Recall that \( s = r^2 \), and since we want the positive value, we have \( r^2 = (1 + \sqrt{5})/2 \). We need to find the ratio \( x/r^2x \), and the \( x \) values cancel, so we get the reciprocal of \( (1 + \sqrt{5})/2 \), which is \( 2/(1 + \sqrt{5}) = 0.62 \).
272. The diagonal of a rectangle divides the rectangle into two right triangles. Using the Pythagorean Theorem, we have $15^2 = x^2 + 9^2$, which simplifies to $225 = x^2 + 81 \rightarrow x^2 = 144$. Taking the square root of both sides yields $x = \sqrt{144} = 12$. Notice this is the 3-4-5 Pythagorean triple multiplied by 3.

273. Using the Pythagorean Theorem, we have $8^2 = x^2 + 5^2$, which simplifies to $64 = x^2 + 25 \rightarrow x^2 = 39$. Taking the square root of both sides yields $x = \sqrt{39} = 6.2$.

274. The altitude drawn from the vertex angle of an isosceles triangle hits the base of the triangle at its midpoint. Therefore the base of this triangle is split into 5 units and 5 units. On the right there is a right triangle with a leg of 5 units and a hypotenuse of 13 units. Using the Pythagorean Theorem, we have $13^2 = x^2 + 5^2$, which simplifies to $169 = x^2 + 25 \rightarrow x^2 = 144$. Taking the square root of both sides yields $x = \sqrt{144} = 12$. Notice this is the 5-12-13 Pythagorean triple.

275. Though not completely drawn in, notice that segment AH is the hypotenuse of right $\triangle ADH$. We know $AH = x$ and $AD = BC = 18$. We need to find DH. Segment DH is the hypotenuse of right $\triangle DCH$ with $DC = EH = 24$ and $CH = 30$. Using the Pythagorean Theorem, we have $DH^2 = 24^2 + 30^2$, which simplifies to $DH^2 = 576 + 900 = 1476$. Since we’re going to plug the value for DH into the Pythagorean Theorem for $\triangle ADH$, let’s not simplify it just yet. For $\triangle ADH$ we have $x^2 = 18^2 + DH^2 \rightarrow x^2 = 18^2 + 1476 \rightarrow x^2 = 324 + 1476 \rightarrow x^2 = 1800$. Taking the square root of both sides yields $x = \sqrt{1800} = 30\sqrt{2} \approx 42.4$.

276. Dropping a vertical line from the point $(x, 40)$, a right triangle is created with a hypotenuse of 41 units and legs of $x$ and 40 units. Using the Pythagorean Theorem, we have $41^2 = x^2 + 40^2$, which simplifies to $1681 = x^2 + 1600 \rightarrow x^2 = 81$. Taking the square root of both sides yields $x = 9$. Notice this is the 9-40-41 Pythagorean triple.

277. The diagonal of a square bisects the square into two right triangles (meaning the angles of each triangle measure 45 degrees, 45 degrees and 90 degrees). This is a special right triangle because there is a relationship between the legs of the triangle and the hypotenuse. The hypotenuse is always equal to a leg times $\sqrt{2}$. Therefore, in this 45-45-90, $x = 5\sqrt{2} \approx 7.1$.

278. When an altitude is dropped in an equilateral triangle, it creates two 30-60-90 triangles. A 30-60-90 triangle is a special right triangle because there is a relationship between the short leg (opposite the 30-degree angle), long leg (opposite the 60-degree angle) and hypotenuse. The hypotenuse is always double the short leg. The long leg is always $\sqrt{3}$ times the short leg. In this case, we have that the long leg is 4\sqrt{3} and we want the hypotenuse. The long leg has a relationship with the short leg. Dividing 4\sqrt{3} by $\sqrt{3}$ will give us our short leg of 4 units. Then we know our hypotenuse is double the short leg or $2 \times 4 = 8$.

279. A regular hexagon can always be divided into six equilateral triangles when opposite vertices of the hexagon are connected. If we label the center of the circle P and the top, horizontal side of the hexagon as segment AB, we can see that $\triangle ABP$ is an equilateral triangle, and $\overline{AP} = \overline{BP} = \overline{AB}$. Using the first two ratios, we get $10\times(10-x) = 8 \times 8$, so $100 - 10x = 64 \rightarrow 10x = -36 \rightarrow x = 3.6$.

Sequences Stretch

283. The common difference between consecutive terms in the sequence is 8. Consequently, the 100th term will be $a + (99 \times 8) = 795$.

284. The sum $S_n$ of the first n terms of an arithmetic sequence is given by the formula $S_n = [n(a_1 + a_n)]/2$, where $a_1$ is the first term and $a_n$ is the nth term. Therefore, the sum of the first 100 terms of this arithmetic sequence is $S_{100} = [100(3 + 795)]/2 = 79800/2 = 39,900$.

285. The common ratio between consecutive terms in the sequence is 1/3. Consequently, the 10th term will be $729 \times (1/3)^9 = 3^9/3^9 = 1/27$.

286. In an arithmetic sequence, successive terms are determined by adding a common difference to the previous term. When we get to the 18th term, $a_{18}$, we would have added the common difference to the first term 17 times. In other words, if $d$ is the common difference between terms of the sequence, then $a_{18} = a_1 + 17d$. We can substitute the given values for the first and 18th terms to get the following equation: $8.25 = 4 + 17d$. Subtracting 4 from each side, we get $4.25 = 17d$, and then dividing each side by 17 yields $d = 0.25$. Now for the 35th term we have $a_{35} = a_1 + 34d$. Substituting 4 for $a_1$ and 0.25 for $d$, we see that $a_{35} = 4 + (34 \times 0.25) = 4 + 8.5 = 12.5$. Also notice that the difference between the 18th and 1st terms, which is 17d, must be the same as the difference between the 35th and 18th terms, also 17d. Since $17d = 4.25$, we know the 35th term is $8.25 + 4.25 = 12.5$. 

88 MATHCOUNTS 2011-2012
287. Consecutive terms in an arithmetic sequence have a common difference. Consequently, the difference between the 1st and 2nd terms must equal the difference between the 2nd and 3rd terms. Hence, 2p + 6 = p = 5p − 12 = (2p + 6). This can be simplified to p + 6 = 3p − 18, which yields 2p = 24, so p = 12. Substituting p = 12 into the expressions for the first three terms shows that the sequence begins 12, 30, 48, ..., with common difference 18. The 4th term must be 48 + 18 = 66.

288. Consecutive terms in a geometric sequence have a common ratio. Consequently, the ratio of the 1st and 2nd terms must equal the ratio of the 2nd and 3rd terms. This fact leads to: \((n + 3)/n = (2n + 6)/(n + 3)\) \(\rightarrow (n + 3)/n = 2 \rightarrow n + 3 = 2n \rightarrow n = 3\). Substituting \(n = 3\) into the expressions for the first three terms shows that the sequence begins 3, 12, ..., with common ratio 2. The 4th term must be \(12 \times 2 = 24\).

289. Let \(d\) represent the common difference between consecutive terms in the sequence. Because 17 is the 3rd term and 83 is the 9th term, \(17 + 6d = 83\) and \(6d = 66\), so \(d = 11\). The 1st term must be \(17 − (2 \times 11) = −5\).

290. Because 24 is the 2nd term and 81 is the 5th term, \(24 \times r^3 = 81\), so \(r = 3/2\). The 1st term must be \(24 \div (3/2) = 16\).

291. In an arithmetic sequence there is a common difference, \(d\), between consecutive terms. Therefore, the fifth term is \(24 − d\) and the seventh term is \(24 + d\). Their sum is \((24 − d) + (24 + d) = 48\).

292. Since 6 hours pass and the number of cells doubles each hour, there will be \(8 \times 2^6 = 512\) cells in the dish at the end of the 8th hour.

**Similarity Stretch**

293. We know that \(\triangle ADE\) is similar to \(\triangle ABC\) since the parallel lines give us \(\angle ADE \cong \angle ABC\) and \(\angle A \congruent\) to itself. Then \(AD:AB = DE:BC\), and letting \(BD = x\), we have \(3/x = 5/20\). If we cross multiply we have the following: \(5 \times (3 + x) = 3 \times 20\). Solving we find that \(5x = 60 \rightarrow 5x = 45 \rightarrow x = 9\).

294. When two figures are similar, any two corresponding linear measurements will be in the same ratio as the sides. From this fact it follows that the perimeters are in the same ratio as the sides. Since the perimeter of the larger pentagon is 26, and since the corresponding sides are in the ratio 2:1, it follows that the perimeter of the smaller pentagon is 13.

295. If we can be convinced that trapezoids RSTU and RSWV are similar, then since their sides would have to be in the ratio of UT:VW or 4:1, and since segment YT in the larger trapezoid corresponds to segment XW in the smaller one, we would know that YT = 40. For two triangles to be similar, it suffices to show that just two pairs of corresponding angles are congruent. But for quadrilaterals, we must establish much more. (Notice that a square and a non-square rectangle have four pairs of corresponding angles congruent, yet the figures are clearly not similar.) For the two trapezoids in question, we know that all four pairs of corresponding angles are congruent because of the parallel lines. We also know that the top bases and the bottom bases are both in the ratio 1:4. What about the legs? If we extend \(RU\) and \(ST\) to intersect at point A, then we can create three similar triangles. If \(AR = c\), then \(AV = 4c\) and \(AU = 16c\). Then \(RV = 3c\) and \(VU = 12c\), and so \(RV:VU = 1:4\). Similarly for segments SW and WT. And so, we can be sure that all pairs of corresponding sides of the two trapezoids are proportional and therefore, YT = 40.

296. Angles formed by two chords of a circle are called inscribed angles, and the measure of an inscribed angle is half the measure of the arc it intersects. As a result, inscribed angles that intersect the same arc of a circle are congruent. In the figure, \(\angle M\) and \(\angle N\) both intersect arc KL, and so they are congruent angles. Likewise, \(\angle K\) and \(\angle L\) are congruent. Thus, \(\triangle NKP\) is similar to \(\triangle MLP\). And \(\triangle NP\) is \(\triangle NK\) and \(\triangle ML\) is \(\triangle NL\). That is, \(\triangle NP = \triangle NK\) and solving gives us \(NP = 4.8\).

297. Since areas of figures come from multiplying measurements in two different dimensions (like the length and width for a rectangle), if two figures are similar with sides in the ratio \(a:b\), then their areas will be in the ratio \(a^2:b^2\). We know that the small triangle and the large triangle in the figure are similar, and their sides are in the ratio 3:5. Thus the areas of the triangles are in the ratio 9:25 and must be 9x and 25x for some value of x. Therefore, the area of the trapezoid is 25x − 9x = 16x. The ratio of the area of the small triangle to the area of the trapezoid is \(9/16\).

298. This time we use the idea of problem 297 in reverse. Since \(\triangle ABE\) and \(\triangle DEC\) are similar, then the ratio of side AB to side DC must be 5 to 7. Since \(\angle ABD\) and \(\angle ACD\) have the same height, their areas must also be in the ratio 25 to 49. The same is true for \(\angle ABC\) and \(\angle BCD\). Let \(u\) denote the area of \(\triangle AED\), and let \(v\) denote the area of \(\triangle BEC\). Then \(u + 5/2 = 7/2 = v + 25/2\), which leads to \(u \approx v \approx 35\). Then the area of trapezoid ABCD is \(25 + 49 + 35 + 35 = 144\).

299. By reasoning much like that used in the solution to problem 297, if two figures are similar with sides in the ratio \(a:b\), then their volumes will be in the ratio \(a^3:b^3\). The corresponding linear measurements in the three cones in the figure are in the ratio 1:2:3, and so the volumes of the three cones are in the ratio 1:8:27. Then it follows that the ratio of the smallest cone to the smaller frustum to the larger frustum is 1:7:19. Therefore, the ratio of the volume of the smaller frustum to the volume of the larger frustum is \(7/19\).

300. This problem combines the ideas of problem 297 and problem 299. Since the volumes are in the ratio 1000 to 216, the sides must be in the ratio of 10 to 6 or 5 to 3. Then the surface areas must be in the ratio 25 to 9. Then letting \(S\) denote the surface area of the smaller house, we have \(25/9 = 400/5\), which leads to a surface area in square centimeters of 144.
Let $a$ be the length of FC. Because of the properties of parallel lines, we have $\triangle EFC \sim \triangle ABC$ and $\triangle EFB \sim \triangle DCB$. Then $\frac{h}{12} = \frac{a}{20}$ and $\frac{h}{6} = \frac{20-a}{20}$. Solving the first for $a$ gives $a = \frac{5}{3}h$, and substituting this into the second equation and solving gives $h = 4$.

We could proceed exactly as in the solution for the previous problem, but a more clever approach is to let $BF = u$ and $FC = v$. Again, $\triangle EFC \sim \triangle ABC$ and $\triangle EFB \sim \triangle DCB$. Then $\frac{h}{a} = \frac{v}{u + v}$ and $\frac{h}{b} = \frac{u}{u + v}$. Adding these two equations together, we get $\frac{h}{a} + \frac{h}{b} = \frac{v}{u + v} + \frac{u}{u + v} = 1$. (Notice that BC was not needed! The height remains the same no matter how far apart the poles are.)

What About Math?

1. If we let $d$ represent the distance to the thunderstorm, we can set up the following proportion: $3 \text{ sec} / 1 \text{ km} = 5 \text{ sec} / d \text{ km}$. Cross-multiplying we get $3d = 5$. Then if we divide each side of the equation by 3, we see that, to the nearest tenth, the thunderstorm is $d = 5/3 \approx 1.7$ km away.

2. Again, we will let $d$ represent the distance to the storm. We have $5 \text{ sec} / 1 \text{ mi} = 12 \text{ sec} / d \text{ mi}$. Cross-multiplying and then dividing each side by 5, we see that the distance to the storm is about $5d = 12 \rightarrow d = 12/5 = 2.4$ mi.

3. The least reasonable value is 1% of 20 km/h, which is $0.01 \times 20 = 0.2$ km/h. The greatest reasonable value is 10% of 20 km/h, which is $0.10 \times 20 = 2$ km/h. Thus the positive difference between the greatest and least reasonable values is $2 - 0.2 = 1.8$ km/h.

4. When the wind speed goes from 50 km/h to 150 km/h it increases by a factor of $150/50 = 3$. That means the wind pressure increases by a factor of $3^2 = 9$.

5. If the ratio of water (melted snow) depth to snowfall depth is 1:10 we have a density of $1/10 = 0.1 \text{ g/cm}^3$. So, if we have a ratio of water depth to snowfall depth of 1:3, the density of the snow is $1/3 = 0.33 \text{ g/cm}^3$.

6. We can solve this problem by setting up a proportion. For freshly fallen snow, the ratio of snow to water is 10 in/1 in. Let $w$ represent the inches of water equivalent to 30 cm of freshly fallen snow. We have $10 \text{ in} / 1 \text{ in} = 30 \text{ cm} / w \text{ cm}$. Cross-multiplying and dividing both sides of the equation by 10, we see that the water equivalent of 30 cm of snow is $10w = 30 \rightarrow w = 3 \text{ cm}$.

7. If the air temperature drops by 3.5 °F every 1000 ft, we need to divide the 18,000 by 1000 to determine how many such increases occur. Since $18,000 \div 1000 = 18$, we see that the outside temperature decreases a total of $18 \times 3.5 = 63$ °F. Since the air temperature was originally 70 °F at sea level, that means the air temperature outside the airplane is $70 - 63 = 7$ °F.

8. We can think of the doubling of 377.6 ppm as an increase of 377.6 ppmv. At a rate of increase of 1.5 ppmv per year, it will take $377.6 \div 1.5 \approx 251.7$ years for the level of CO$_2$ to double.

9. If the rate of increase were 3.0 ppmv per year, that would be twice as fast at the rate of 1.5 ppmv per year. That means the level of CO$_2$ would double in half the time or $251.7 \div 2 = 125.9$ years.

10. If 15% of the radiation is absorbed and 20% is scattered, that accounts for a total of 15% + 20% = 35% of the radiation. That leaves the remaining 100% - 35% = 65% of the radiation to fall to Earth’s surface.